4.4 Permutations When Objects Are Identical

YOU WILL NEED

- calculator

EXPLORE...

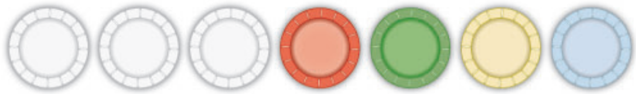
- Consider the words BIKE and BOOK. How does the number of ways that the four letters in each word can be arranged differ? Why?

GOAL

Determine the number of permutations when some objects are identical.

INVESTIGATE the Math

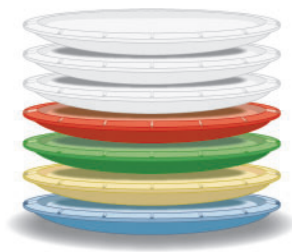
Tom is putting away the dishes after supper. In addition to putting away the bowls, cups, and cutlery, he has to stack seven dinner plates. Three of the plates are white and identical, while the remaining four plates are red, green, yellow, and blue.



? How many different ways can he stack the seven plates to store them in the cupboard?

A. If all seven plates are different, in how many ways could he stack the plates?

B. Tom decided to think about the plates as if they are different. To do this, he represented the three identical white plates using three different letter codes: W_1, W_2, and W_3. He then used R for the red plate, G for the green plate, Y for the yellow plate, and B for the blue plate. List all the ways the plates could be stacked if the three white plates are stacked on top of the four coloured plates, and the four coloured plates are stacked in the order red, green, yellow, blue.



C. Examine your list in part B. Recognizing that the white plates are, in fact, identical, how many arrangements in your list are really the same arrangement? How does this number relate to the number of white plates?

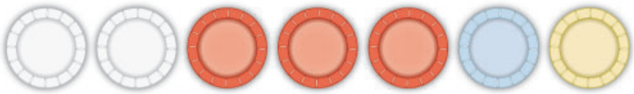
D. Use your answers for parts A and C to write an expression that represents the number of different ways these seven plates can be stacked. Use your expression to calculate the number.

Reflecting

E. Suppose Tom had to stack these plates:

Write an expression to represent the number of different ways to stack the plates.

F. Suppose Tom had to stack these plates:



Write an expression to represent the number of different ways to stack the plates.

G. The next time Tom puts away the dishes, the number of arrangements that are possible when he stacks the plates is represented by the expression

$$\frac{10!}{2! \cdot 3! \cdot 4!}$$

How many plates will he be stacking and what colours might they be?

H. Write an expression to represent the number of permutations of n objects, where a of the objects are identical, another b are identical, and another c are identical.

APPLY the Math

EXAMPLE 1 Solving a permutation problem where objects are alike

In the mountainous regions of India, China, Nepal, and Bhutan, it is common to see prayer flags. Each flag has a prayer written on it, and colour is used to symbolize different elements: green (water), yellow (earth), white (air/wind), blue (sky/space), and red (fire).

How many different arrangements of the same prayer can Dorji make using these 9 flags: 1 green, 1 yellow, 2 white, 3 blue, and 2 red?

Dorji's Solution

Let A represent the number of arrangements of 9 flags:

$$A = \frac{9!}{2! \cdot 3! \cdot 2!}$$

If the 9 flags were different, I could arrange them 9! ways. However, some are identical. Since there are 2 white flags, 3 blue flags, and 2 red flags, I divided 9! by 2! · 3! · 2! to eliminate the arrangements that are the same and would be counted more than once otherwise. For example, if I didn't divide by 2! for the 2 identical white flags, these arrangements would be counted as two different arrangements instead of one:

$G\ Y\ B_1\ B_2\ B_3\ W_1\ W_2\ R_1\ R_2$
$G\ Y\ B_1\ B_2\ B_3\ W_2\ W_1\ R_1\ R_2$

$$A = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3!}{2 \cdot 1 \cdot 3! \cdot 2 \cdot 1}$$

$$A = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5$$

$$A = 15\ 120$$

I simplified using the facts that $\frac{3!}{3!} = 1$ and $\frac{4}{2 \cdot 1 \cdot 2 \cdot 1} = 1$

There are 15 120 different prayer flag arrangements.

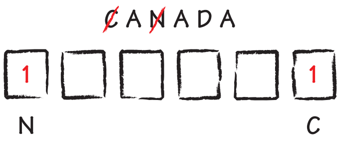
Your Turn

Suppose there are 9 flags, but 3 are white, 3 are red, and 3 are green. Predict whether there would be more or fewer than 15 120 arrangements of these flags. State any assumptions you are making. Verify your prediction by determining the number of arrangements.

EXAMPLE 2 Solving a conditional permutation problem involving identical objects

How many ways can the letters of the word CANADA be arranged, if the first letter must be N and the last letter must be C?

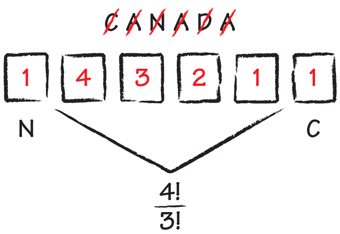
Sung Ho's Solution



I drew a box diagram to help me organize my solution.

I knew that each arrangement would contain six letters, and there are two conditions to be met: the first letter in each arrangement had to be N and the last letter C.

Since there is only one of each of these letters, these positions could be filled only one way.



That left four letters left to place. I knew these could be arranged in 4! ways if they are all different, but they aren't. The three A's are identical.

I divided 4! by 3! to eliminate the arrangements that would be the same.

Let A represent the number of arrangements:

$$A = 1 \cdot \frac{4!}{3!} \cdot 1$$

$$A = 1 \cdot 4 \cdot 1$$

$$A = 4$$

I calculated the total number of arrangements using the Fundamental Counting Principle because I am placing the N in the first position AND placing the C in the last position AND arranging the remaining four letters in the four positions in between.

There are four ways the letters in CANADA can be arranged with N as the first and C as the last letter.

NDAAAC

NADAAC

NAADAC

NAAADC

I created an organized list to check my answer.

Since the N and C must be in the first and last positions, I knew the three A's and the D must be in between. The only way I could create different arrangements was to change the position of the D.

My answer seems reasonable, since I was able to list only four different possibilities.

Your Turn

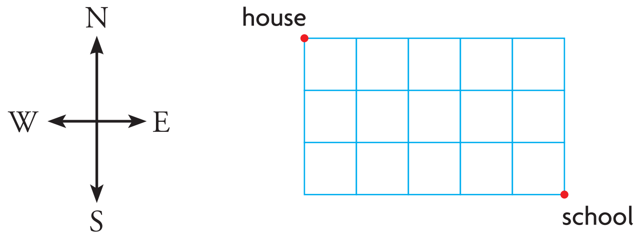
How many different arrangements are there for the six letters in the word CANADA for each situation?

a) If there are no conditions for where letters must be placed

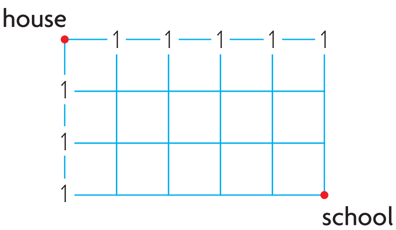
b) If the first letter has to be C

EXAMPLE 3 Solving a permutation problem involving routes

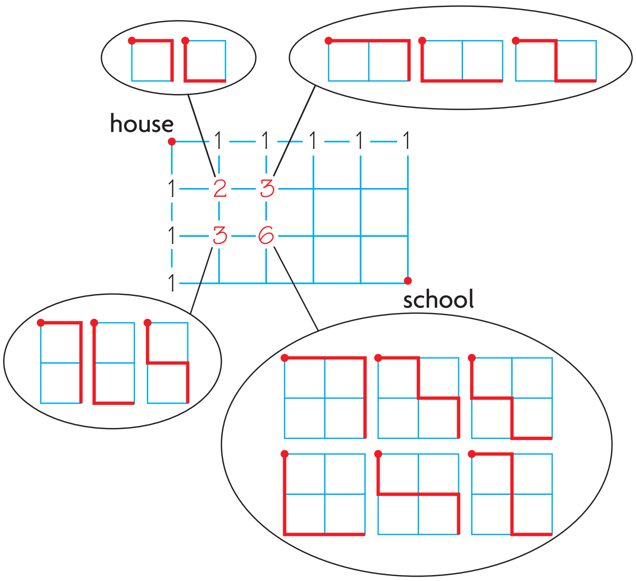
Julie's home is three blocks north and five blocks west of her school. How many routes can Julie take from home to school if she always travels either south or east?



Julie's Solution: Using a diagram



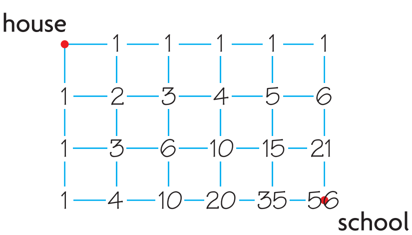
I drew a diagram that indicated the number of ways I could get to each intersection along the perimeter by going just south or just east.



Then I started to count the number of ways I could get to other intersections by going both south and east.

I noticed a pattern; the sum of the numbers on the top right and bottom left corners of each block is equal to the number of routes to the bottom right corner of each block.

$1 + 1 = 2$, $1 + 2 = 3$, $2 + 1 = 3$, and $3 + 3 = 6$

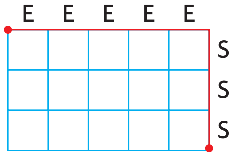


Using the pattern, I completed the diagram.

There are 56 different routes from home to school travelling south or east.

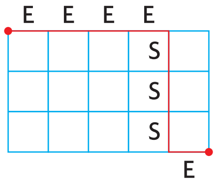
Jean's Solution: Using permutations

Possible routes:



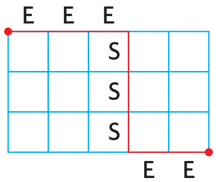
EEEEESSS

Since her school is five blocks east and three blocks south of her home, any route she takes will involve travelling east five blocks and south three blocks.



EEEESSSE

I started to create a list of possible routes using the letter E for east and S for south.



EEESSSEE

I stopped when I realized that this is just a permutation problem involving identical objects. I decided to figure out how many different arrangements there are for the five E's and three S's.

Let R represent the number of routes:

$$R = \frac{8!}{5! \cdot 3!}$$

To determine the number of routes, I needed to determine the number of permutations of eight letters, where five are E's and three are S's.

$$R = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{5! \cdot 3!}$$

$$R = 8 \cdot 7$$

$$R = 56$$

I simplified using the facts that

$$\frac{5!}{5!} = 1 \text{ and } \frac{6}{3!} = 1$$

There are 56 different routes from home to school travelling south or east.

Your Turn

The school is three blocks east and four blocks south of Carrie's house. Predict whether Carrie will have more or fewer than 56 possible routes if she always travels south or east. Determine the number of routes to verify.

In Summary

Key Ideas

- There are fewer permutations when some of the objects in a set are identical compared to when all the objects in a set are different. This is because some of the arrangements are identical.
- The number of permutations of n objects, where a are identical, another b are identical, another c are identical, and so on, is

$$P = \frac{n!}{a!b!c!...}$$

For example, in the set of four objects a, a, b, and b, the number of different permutations, P, is

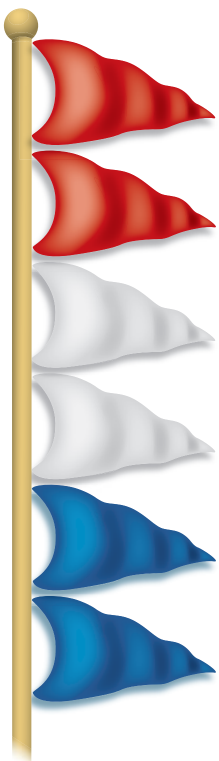
$$P = \frac{4!}{2! \cdot 2!}$$

$$P = 6$$

The six different arrangements are *aabb*, *bbaa*, *abab*, *baba*, *abba*, and *baab*.

Need to Know

- Dividing $n!$ by $a!$, $b!$, $c!$, and so on deals with the effect of repetition caused by objects in the set that are identical. It eliminates arrangements that are the same and that would otherwise be counted multiple times.



CHECK Your Understanding

1. Evaluate the following expressions.

a) $\frac{7!}{3! \cdot 2!}$ **b)** $\frac{8!}{2! \cdot 2! \cdot 2!}$ **c)** $\frac{10!}{4! \cdot 3! \cdot 2!}$ **d)** $\frac{12!}{2! \cdot 4! \cdot 5!}$

2. How many different signals can be made from 6 flags hung in a vertical line, using 2 identical white flags, 2 identical red flags, and 2 identical blue flags?

3. Six nickels are flipped simultaneously. How many ways can three coins land as heads and three coins land as tails?

PRACTISING

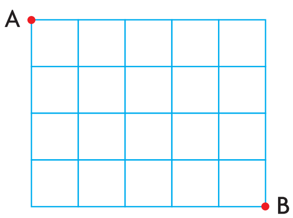
4. A hockey team has a record of 10 wins, 5 losses, and 3 ties in 18 games. How many different ways could this record have occurred?

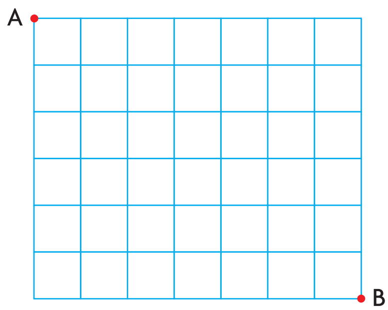
5. Norm bought 3 chocolate-chip cookies, 2 peanut-butter cookies, and 4 oatmeal cookies from the corner bakery to give to his 9 grandchildren. How many ways can he distribute 1 cookie to each grandchild?

6. How many different arrangements can be made using all the letters in each word?
   a) YUKON
   b) ALBERTA
   c) MANITOBA
   d) SASKATCHEWAN

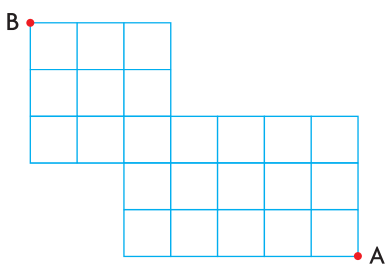
7. A clerk at a bookstore is restocking a shelf of best-selling novels. He has five copies each of three different novels.
   a) How many different ways can he arrange the books on the shelf?
   b) How many different ways can these books be arranged on the shelf if the copies of the same novel must be grouped together?

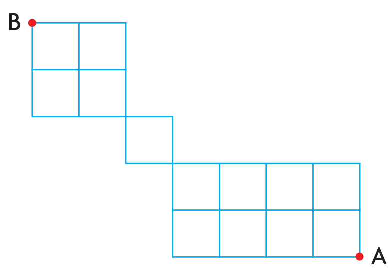
8. Create a counting problem that can be solved using the expression $\frac{8!}{2! \cdot 4!}$.

9. Determine the number of routes there are to get from point A to point B, if you travel only south or east.
   a)


   b)


10. Jess always walks to her friend's house, which is eight blocks north and five blocks west of her house. How many different routes can she take if she always walks either north or west?

11. How many different routes are there from A to B, if you travel only north or west?
   a)


   b)


12. A true–false test has eight questions. How many different permutations of answers can the teacher create if five answers are true and three answers are false?

13. The Chief Wakas Totem Pole, located in Vancouver's Stanley Park, represents the talking stick and characters in an Owikeno (Kwakiutl) story. In the photograph below, the Chief Wakas Totem Pole is on the right. The figures, from the top down, are Thunderbird, Killer Whale, Wolf, Wise One, Huxwhukw (a mythical bird), Bear, and Raven.

The original Chief Wakas Totem Pole (a replica is pictured, far right) was built as a ceremonial entrance to Chief Wakas's house in Alert Bay, Vancouver Island.

a) Suppose a new totem pole is created using different permutations of all seven figures. How many different arrangements of these figures on the pole are possible?

b) Suppose another new totem pole is to include these seven figures: two of Thunderbird and two of Wolf, as well as Killer Whale, Wise One, and Bear. How many ways can these figures be arranged on a totem pole?

14. Explain why ${}_nP_n = n!$ cannot be used to determine the number of arrangements of a group of n items when there are a identical items in the group and $a < n$.

15. a) How many ways can 9 coins (3 dimes, 4 quarters, and 2 loonies) be arranged in a line? State any assumptions you are making.

b) What if the line must begin and end with a loonie? State any assumptions you are making.

16. Sam, Nunzio, and 8 of their friends are playing outside on a hot summer day. How many ways can 10 freezies (3 grape, 2 lime, and 5 orange) be distributed among the 10 children if Sam must have lime and Nunzio must have grape?

17. How many permutations are possible using all the letters of the word STATISTICS for each condition described below?

a) You must start with A and end with C.

b) The two I's must be together.

Closing

18. Consider the words BANANAS and BANDITS. Explain why there are 12 times the number of permutations possible using all the letters of BANDITS compared to the number of permutations possible using all the letters of BANANAS.

Extending

19. A Rubik's cube can be thought of as a grid in three dimensions. How many routes are there from the top rear vertex of the cube to the lower front vertex of the cube, if each route must be as short as possible and follow the grid lines?

20. How many ways can 20 soccer players on a travelling team be assigned to hotel rooms for each situation?

a) There are only 10 double rooms.

b) There are 5 quadruple rooms.

21. A bag contains three identical red marbles and three identical white marbles. Four marbles are drawn out of the bag and arranged in a row from left to right.

a) How many different arrangements might be made?

b) What is the likelihood that the arrangement is, from left to right, red, white, white, red?