4.6 Combinations

GOAL

Solve problems involving combinations.

YOU WILL NEED

- calculator
- standard deck of playing cards

EXPLORE...

- Five cards are dealt to each person in a card game. How many ways can you be dealt a hand that has only red cards?

LEARN ABOUT the Math

Each year during the Festival du Voyageur, held during February in Winnipeg, Manitoba, high schools compete in the Voyageur Snow Sculpture Contest. This year Amir's school will enter a three-person team. Nine students have volunteered to be on the team.

? In how many ways can a team of three snow sculptors be chosen to represent Amir's school from the nine students who have volunteered?

EXAMPLE 1 Calculating combinations

Determine the number of three-person teams that can be formed from the nine volunteers.

Amir's Solution

	First Choice	Second Choice	Third Choice
Team A	Bill	Amir	Connie
Team B	Bill	Connie	Amir
Team C	Amir	Bill	Connie
Team D	Amir	Connie	Bill
Team E	Connie	Bill	Amir
Team F	Connie	Amir	Bill

I started by trying to list all the possible three-person teams. I started by just considering three of the nine possible students. I realized that teams A to F are really the same team. In this situation, the order in which the members of the team are chosen doesn't matter.

$${}_nP_r = \frac{n!}{(n-r)!}$$

where $n = 9$ and $r = 3$

$${}_9P_3 = \frac{9!}{(9-3)!}$$

$${}_9P_3 = \frac{9!}{6!}$$

$${}_9P_3 = \frac{9 \cdot 8 \cdot 7 \cdot 6!}{6!}$$

$${}_9P_3 = 9 \cdot 8 \cdot 7$$

$${}_9P_3 = 504$$

I knew I could calculate the number of permutations involving three people from nine people using the permutation formula.

But 504 is the number of teams when order matters. In this situation, I want to consider only the number of combinations.

Therefore, 504 teams of three can be formed when the order in which they are chosen matters.

$${}_3P_3 = 3!$$

$${}_3P_3 = 3 \cdot 2 \cdot 1$$

$${}_3P_3 = 6$$

The teams listed on the previous page involving Bill, Amir, and Connie are the same because the order in which each team member is selected doesn't matter. These three students can be arranged in 3! or 6 different ways. This is true regardless of which three students are chosen.

$$\frac{{}_9P_3}{{}_3P_3} = \frac{504}{6}$$

$$\frac{{}_9P_3}{{}_3P_3} = 84$$

I divided the number of permutations, 504, by the number of arrangements that are the same, 6. This gave me the number of combinations.

Therefore, 84 different teams of three can be formed.

Reflecting

A. Why was it necessary to divide ${}_9P_3$ by 3! ?

B. Express $\frac{{}_9P_3}{{}_3P_3}$ in terms of factorials.

C. Hanna claims that to determine the number of combinations of r objects chosen from a set of n different objects, you divide ${}_nP_r$ by r!. Do you agree? Explain.

D. Write a formula you could use to determine ${}_nC_r$, representing the number of combinations possible in which r objects are chosen from a set of n different objects.

Communication | *Notation*

${}_nC_r$ or $\binom{n}{r}$ are notations commonly used to represent the number of combinations that can be made from a set of n different objects where only r of them are used in each grouping, and $0 \le r \le n$.

${}_nC_r$ and $\binom{n}{r}$ are read as "n choose r."

APPLY the Math

EXAMPLE 2 Solving a simple combination problem

A restaurant serves 10 flavours of ice cream. Danielle has ordered a large sundae with three scoops of ice cream. How many different ice-cream combinations does Danielle have to choose from, if she wants each scoop to be a different flavour?

Danielle's Solution

Each of these is the same sundae:

V C S　　S C V

C V S　　S V C

V S C　　C S V

I started to visualize all the ways the scoops could be placed in the dish and then realized that the position of each scoop in the dish doesn't matter. For example, one scoop of vanilla, one scoop of chocolate, and one scoop of strawberry is the same sundae no matter how the scoops are arranged.

$${}_nC_r = \frac{n!}{r!(n-r)!}$$

where $n = 10$ and $r = 3$

Since order doesn't matter, I knew I needed to determine the number of combinations using only 3 flavours out of a possible 10.

$${}_{10}C_3 = \frac{10!}{3!(10-3)!}$$

$${}_{10}C_3 = \frac{10!}{3! \cdot 7!}$$

$${}_{10}C_3 = \frac{10 \cdot 9 \cdot 8 \cdot 7!}{3 \cdot 2 \cdot 1 \cdot 7!}$$

I simplified by dividing both the numerator and denominator by 7!.

$${}_{10}C_3 = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1}$$

I divided 9 by 3 and 8 by 2.

$${}_{10}C_3 = 10 \cdot 3 \cdot 4$$

$${}_{10}C_3 = 120$$

There are 120 different three-scoop sundaes with a different flavour for each scoop.

Your Turn

Danielle's favourite flavour is chocolate. If one scoop in her large sundae must be chocolate and the other two must be different flavours, how many combinations of ice cream are possible?

EXAMPLE 3 Solving a combination problem using the Fundamental Counting Principle

Tanya is the coach of a Pole Push team that consists of nine players: five male and four female. In each competition, teams of four compete against each other to push their competitors out of a circle. The team that is successful wins.

a) How many different four-person teams does Tanya have to choose from for an all-male competition?

b) How many different four-person teams does Tanya have to choose from, with two males and two females, for a mixed competition?

Pole Push is played during the Arctic Winter Games. This team from the Northwest Territories placed second in the junior female competition at the 2006 Games, held in Kenai, Alaska.

Tanya's Solution

a) $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

where $n = 5$ and $r = 4$

$$\binom{5}{4} = \frac{5!}{4!(5-4)!}$$

$$\binom{5}{4} = \frac{5!}{4! \cdot 1!}$$

$$\binom{5}{4} = \frac{5 \cdot 4!}{4!}$$

$$\binom{5}{4} = 5$$

There are five different all-male teams that can be chosen from the nine players.

I knew that I needed to select four men from the five who were available. Since I was selecting a team of four men, and there were no specific positions within each team, I knew that order doesn't matter. That meant I could use the combination formula.

b) Men:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

where $n = 5$ and $r = 2$

Women:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

where $n = 4$ and $r = 2$

I knew that I needed to select two men from the five who were available and two women from the four who were available. Since I was selecting the team and not assigning positions, order did not matter, making this a combination problem.

Let T represent the number of teams:

$$T = \binom{5}{2}\binom{4}{2}$$

$$T = \frac{5!}{2!(5-2)!} \cdot \frac{4!}{2!(4-2)!}$$

$$T = \frac{5!}{2! \cdot 3!} \cdot \frac{4!}{2! \cdot 2!}$$

$$T = \frac{5 \cdot 4 \cdot 3!}{2! \cdot 3!} \cdot \frac{4 \cdot 3 \cdot 2!}{2! \cdot 2!}$$

$$T = 10 \cdot 6$$

$$T = 60$$

Since I was selecting men AND selecting women to create the team of four, I multiplied the two combinations using the Fundamental Counting Principle.

There are 60 different four-person teams of two men and two women that can be chosen from the nine players.

Your Turn

How many different four-person teams does Tanya have to choose from for an all-women competition?

EXAMPLE 4 Solving a combination problem by considering cases

A planning committee is to be formed for a school-wide Earth Day program. There are 13 volunteers: 8 teachers and 5 students. How many ways can the principal choose a 4-person committee that has at least 1 teacher?

Jarrod's Solution: Using direct reasoning

Case 1: 1 teacher and 3 students

$${}_8C_1 \cdot {}_5C_3 = \frac{8!}{1! \cdot 7!} \cdot \frac{5!}{3! \cdot 2!}, \text{ or } 80$$

Order does not matter when choosing people for a committee that has no assigned positions, so I knew this problem involves combinations.

I considered the different makeup of the possible 4-person committees, including at least 1 teacher each time, as four different cases.

I evaluated the expression for each case using a calculator.

I used direct reasoning to solve the problem. I dealt directly with each of the cases meeting the "at least 1 teacher" condition.

Case 2: 2 teachers and 2 students

$${}_8C_2 \cdot {}_5C_2 = \frac{8!}{2! \cdot 6!} \cdot \frac{5!}{2! \cdot 3!}, \text{ or } 280$$

Case 3: 3 teachers and 1 student

$${}_8C_3 \cdot {}_5C_1 = \frac{8!}{3! \cdot 5!} \cdot \frac{5!}{1! \cdot 4!}, \text{ or } 280$$

Case 4: 4 teachers and 0 students

$${}_8C_4 \cdot {}_5C_0 = \frac{8!}{4! \cdot 4!} + \frac{5!}{0! \cdot 5!}, \text{ or } 70$$

Number of committees $= 80 + 280 + 280 + 70$
Number of committees $= 710$

Since Case 1 OR Case 2 OR Case 3 OR Case 4 are all possible but are exclusive, I added the number of combinations for all four cases.

The principal can choose from 710 different Earth-Day committees that include at least 1 teacher.

Shelby's Solution: Using indirect reasoning

Number of committees with no conditions:

$$\binom{13}{4} = \frac{13!}{4! \cdot 9!}$$

I knew that if I figured out how many 4-person committees are possible from a group of 13 with no conditions, then I could just subtract the number of 4-person committees that have 0 teachers (which is the same as the number of committees with 4 students). The difference will be the committees that have at least 1 teacher.

I used indirect reasoning to solve the problem. I did not deal directly with the cases that meet the "at least 1 teacher" condition. I dealt with the other case instead and subtracted.

Number of committees with 0 teachers and 4 students:

$$\binom{8}{0}\binom{5}{4} = \frac{8!}{0! \cdot 8!} \cdot \frac{5!}{4! \cdot 1!}$$

Number of committees with at least 1 teacher:

$$\binom{13}{4} - \binom{8}{0}\binom{5}{4} = 710$$

The principal can choose from 710 different Earth Day committees that include at least 1 teacher.

Your Turn

a) Compare Jarrod's and Shelby's solutions. Which solution do you think is more efficient? Explain.

b) Would you use direct reasoning or indirect reasoning to solve the following problem: How many ways can the principal choose a four-person committee that has at least one student?

In Summary

Key Ideas

- You can solve counting problems where order is not important by calculating the number of combinations.
- The number of combinations from a set of n different objects, where only r of them are used in each combination, can be denoted by ${}_nC_r$ or $\binom{n}{r}$ and is calculated using the formula

$${}_nC_r = \frac{n!}{r!(n-r)!}, \text{ where } 0 \leq r \leq n$$

For example, if you have a set of three objects, a, b, and c, but you use only two in each combination, the number of combinations is

$${}_3C_2 = \frac{3!}{2!(3-2)!} \text{ or } 3$$

Need to Know

- The formula for ${}_nC_r$ is the formula for ${}_nP_r$ divided by $r!$. Dividing by $r!$ eliminates the counting of the same combination of r objects arranged in different orders.
- When solving problems involving combinations, it may also be necessary to use the Fundamental Counting Principle.
- Sometimes combination problems can be solved using direct reasoning. This occurs when there are conditions involved. To do this, follow the steps below:
  1. Consider only cases that reflect the conditions.
  2. Determine the number of combinations for each case.
  3. Add the results of step 2 to determine the total number of combinations.
- Sometimes combination problems that have conditions can be solved using indirect reasoning. To do this, follow these steps:
  1. Determine the number of combinations without any conditions.
  2. Consider only cases that do not meet the conditions.
  3. Determine the number of combinations for each case identified in step 2.
  4. Subtract the results of step 3 from step 1.

CHECK Your Understanding

1. Joe has a choice of four flavours of ice cream for his two-scoop sundae: vanilla, strawberry, chocolate, and butterscotch.
   a) List all the permutations for a two-flavour sundae.
   b) List all the combinations for a two-flavour sundae.
   c) How is the number of two-flavour permutations related to the number of two-flavour combinations? Explain.

2. From a group of five students, three students need to be chosen for a car-wash committee.
   a) How many committees are possible?
   b) How many committees are possible, if only two students are needed on the committee?
   c) Compare your answers for parts a) and b). What do you notice? Explain why this occurred.

3. How many ways can 6 people be selected from a group of 12 to form a dodge-ball team?

PRACTISING

4. Evaluate the following.
   a) ${}_5C_3$
   b) ${}_9C_8$
   c) $\binom{6}{4}$
   d) ${}_{10}C_0$
   e) $\binom{12}{6}$
   f) ${}_8C_1$

5. How many ways can 6 players be chosen to start a volleyball game from a team of 10?

6. An online music store offers 5 free songs when you join. It has 55 hip-hop songs available. How many different combinations of hip-hop songs can you download for free?

7. The card game Crazy Eights is played with a standard deck of playing cards. How many different 8-card hands can be dealt?

8. Connie's softball team has 15 players. How many ways can the coach choose his starting lineup of 9 players, if Connie must be the pitcher?
   a) Does this problem involve permutations or combinations? Explain.
   b) Solve the problem.

9. a) Marnie claims that $\binom{6}{2} = \binom{6}{4}$. Do you agree? Justify your decision.
   b) Examine several more cases with the same relationship as part a). What do you notice?
   c) Based on your observations in part b), suggest a relationship between $\binom{n}{r}$ and $\binom{n}{n-r}$ for the natural numbers n and whole numbers r.

10. Suppose that 5 teachers and 8 students volunteered to be on a graduation committee. The committee must consist of 2 teachers and 3 students. How many different graduation committees does the principal have to choose from?

11. How many 5-person committees can be formed from a group of 6 women and 4 men, under each of the following conditions.

a) There are no conditions.

b) There must be exactly 3 women.

c) There must be exactly 4 men.

d) There can be no men.

e) There must be at least 3 men.

12. A youth hostel has 3 rooms that contain 5, 4, and 3 beds, respectively. How many ways can 12 students be assigned to these rooms?

13. a) For each expression, state the number of different objects in the set and how many are used in each combination.

i) ${}_nC_r = \dfrac{5!}{3!(5-3)!}$ **ii)** ${}_nC_r = \dfrac{10!}{2!(10-2)!}$ **iii)** ${}_nC_r = \dfrac{5!}{3!2!}$

b) Choose one expression from part a) and create a combination problem that could be solved using that expression.

14. Pascal's triangle can be created using the combinations below.

a) Evaluate each combination.

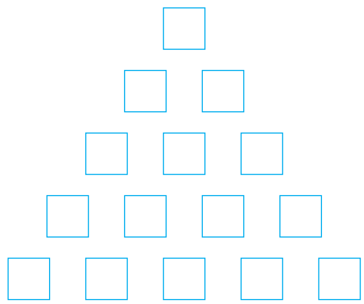
i) ${}_0C_0$

ii) ${}_1C_0$, ${}_1C_1$

iii) ${}_2C_0$, ${}_2C_1$, ${}_2C_2$

iv) ${}_3C_0$, ${}_3C_1$, ${}_3C_2$, ${}_3C_3$

v) ${}_4C_0$, ${}_4C_1$, ${}_4C_2$, ${}_4C_3$, ${}_4C_4$



b) Copy the triangular arrangement of boxes at right. Write each answer from part a) in a box in order, starting with the answer to part i) at the top.

c) In parts a) and b), you created the first five rows of Pascal's triangle. Describe at least two patterns you observe in the triangle.

d) Write two more rows of Pascal's triangle using the patterns you observed.

e) How does Pascal's triangle relate to the pathway problems in Lesson 4.4?

15. Solve each equation. State any restrictions on the variable.

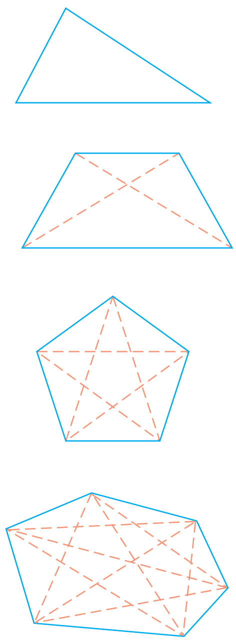
a) ${}_nC_2 = 15$

b) ${}_nC_4 = 35$

c) $4\binom{n}{2} = \binom{n+2}{3}$

d) $\binom{6}{r} = 15$

16. A children's hospital in a city of about one million people is running a charity lottery called Lucky Six to raise money. Players choose six numbers from the numbers 1 to 66. The player wins if the six numbers chosen match six numbers drawn at random by the organizers.
    a) How many ways could the player win?
    b) How many ways could the player lose?
    c) Is this a reasonable game for the hospital to run? Explain.



17. How can the combination formula be used to determine the number of diagonals in an n-sided polygon?

18. There are 7 boys and 13 girls in the school art club. A group of 5 is needed to set up an art exhibit. How many different groups of 5 students with at least 2 boys are there to choose from?
    a) Solve the problem using direct reasoning.
    b) Solve the problem using indirect reasoning.
    c) Which approach do you prefer? Explain why.

Closing

19. a) How are combinations and permutations similar? How are they different? Use examples in your answers.
    b) If you know the value of ${}_nP_r$, how can you determine the value of ${}_nC_r$? Use examples in your answer.

Extending

20. A CD player holds five different CDs. The CD player is set on shuffle so it randomly selects songs to play from the five CDs. This chart shows the number of songs there are on each CD.

CD Number	1	2	3	4	5
Number of Songs	12	14	15	12	18

Determine the probability that each of the following events will happen during the first five songs played.
    a) The five songs will be from CD 2 and CD 4.
    b) One of the five songs will be from each CD.
    c) Your favourite song from each of the 5 CDs will be played.

21. Simplify: $\binom{n}{3} + \binom{n}{2} + \binom{n}{1}$

22. Prove: ${}_{n+1}C_r = {}_nC_r + {}_nC_{r-1}$