4.7 Solving Counting Problems

GOAL

Solve counting problems that involve permutations and combinations.

YOU WILL NEED

- calculator
- standard deck of playing cards

INVESTIGATE the Math

A band has recorded 3 hit singles over its career. One of the hits went platinum. The band members selected 15 songs, including the 3 hits, for the set list of their upcoming concert.

? How many possible sequences for the set list are there if the band wants to open with a hit, play a hit in the middle of the show, and end the concert with their platinum hit?

A. Does the order of the songs matter in each possible set list? Explain.

B. Are there any conditions that have to be met? Explain.

C. How many platinum hit songs are there to choose from for the song in the 15th position?

D. How many hit songs are there to choose from for the song in the 1st position?

E. Once they have chosen the hit song for the 1st position, how many hit songs are there to choose from for the song in the middle (8th) position?

F. How many ways can they arrange the remaining songs around their hits?

G. Write an expression that represents the number of different set lists that meet all the conditions. Evaluate your expression.

EXPLORE...

- There are five swimmers in the first heat of a race: Aubrey, Betz, Cam, Deanna, and Elena.
  a) How many ways can the five swimmers finish first, second, and third?
  b) How many ways can the five swimmers qualify for the final race if the top three finishers qualify?

Reflecting

H. Did your expression in part G involve combinations or permutations?

I. Did you use the Fundamental Counting Principle to create your expression in part G? Explain.

J. Suppose the band had only two hits and wanted to play one at the beginning and one at the end of the concert. Would this increase or decrease the number of possible set lists? By what factor would it change the number of possible set lists?

APPLY the Math

EXAMPLE 1 Solving a permutation problem with conditions

A piano teacher and her students are having a group photograph taken. There are three boys and five girls. The photographer wants the boys to sit together and the girls to sit together for one of the poses. How many ways can the students and teacher sit in a row of nine chairs for this pose?

Yvette's Solution

teacher	boys	girls

In this problem, I knew that order matters, since sitting in a different order creates a different arrangement, or permutation.
I drew a diagram with three boxes to help me visualize the problem: one box for the teacher, one box for the boys, and one box for the girls. I did this to represent all the boys sitting together and all the girls sitting together.

teacher	boys	girls
1	3!	5!

I knew that if I decided to place the teacher in the first box, this can happen in only one way.
In the box for boys, I knew there are 3! permutations. In the box for girls, I knew there are 5! permutations.

T	B	G

T	G	B

B	G	T

G	B	T

B	T	G

G	T	B

Then I realized that the three boxes themselves could be arranged in 3! or 6 ways.

Let S represent the number of seating arrangements:

$S = (1 \cdot 3! \cdot 5!) \cdot 3!$

$S = 1 \cdot 6 \cdot 120 \cdot 6$

$S = 4320$

There are 4320 different ways the students and teacher can sit in a single row of nine chairs.

I used the Fundamental Counting Principle to write an expression to represent all the possible seating arrangements, since we are arranging the boys within one box AND arranging the girls within one box AND arranging the position of the three boxes that represent the teacher, the boys, and the girls.

Your Turn

For another pose, the photographer wants the two tallest students, Jill and Sam, to sit at either end, Jill on the left and Sam on the right, and the teacher to sit in the middle. How many different seating arrangements are there for this pose?

EXAMPLE 2 Solving a combination problem involving multiple choices

Combination problems are common in computer science. Suppose there is a set of 10 different data items represented by {a, b, c, d, e, f, g, h, i, j} to be placed into four different memory cells in a computer. Only 3 data items are to be placed in the first cell, 4 data items in the second cell, 2 data items in the third cell, and 1 data item in the last cell. How many ways can the 10 data items be placed in the four memory cells?

Ariston's Solution

3 of 10 items in cell 1: $\binom{10}{3}$

Since the order in which the items are placed within each memory cell does not matter, I knew that the problem involves combinations.
I started by thinking about the first cell and how many ways 3 of 10 items could be placed there.

4 of the remaining 7 items in cell 2: $\binom{7}{4}$

That left 7 items, of which 4 had to go in the second cell.

2 of the remaining 3 items in cell 3: $\binom{3}{2}$

Of the 3 items remaining, 2 had to go in the third cell.

1 item in cell 4: $\binom{1}{1}$

That left 1 item for the last cell.

Let M represent the number of ways to place all 10 items in cells 1, 2, 3, and 4:

$$M = \binom{10}{3}\binom{7}{4}\binom{3}{2}\binom{1}{1}$$

$$M = 12\ 600$$

Since all 10 data items must be placed in the four memory cells (3 items in cell 1 AND 4 items in cell 2 AND 2 items in cell 3 AND 1 item in cell 4), I used the Fundamental Counting Principle.
I used a calculator, since there are several expressions involved.

The 10 data items can be placed in the four memory cells 12 600 ways.

Your Turn

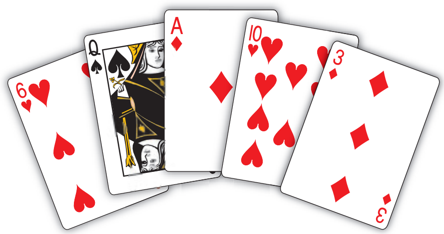
Jim claims that he solved the problem by assigning a cell to each data item. He knew that cell 1 was to get 3 items, cell 2 was to get 4 items, cell 3 was to get 2 items, and cell 4 was to get 1 item. He started to list the possibilities in a chart:

Data Item	a	b	c	d	e	f	g	h	i	j
Cell #	1	1	1	2	2	2	2	3	3	4
Cell #	1	2	3	4	1	2	3	1	2	2
Cell #										

He then realized that each assignment of cells to items corresponded to a different arrangement of the cell numbers. In other words, it was a permutation problem for the numbers 1112222334. Does Jim's line of reasoning result in the same answer?

EXAMPLE 3 Solving a combination problem involving cases

How many different five-card hands that contain at most one black card can be dealt to one person from a standard deck of playing cards?



Sunny's Solution

Case 1: exactly 1 black and 4 red cards
Case 2: exactly 0 black and 5 red cards

Since the order of the cards in each hand does not matter, I knew that the problem involves combinations.
To satisfy the condition that the hand contains at most 1 black card, I separated the situation into the only two possible cases.

Case 1: exactly 1 black and 4 red

${}_{26}C_1 \cdot {}_{26}C_4$

I used the fact that there are 26 black cards and 26 red cards to represent the number of combinations that result in 1 black AND 4 red cards.

Case 2: exactly 0 black and 5 red

${}_{26}C_0 \cdot {}_{26}C_5$

I represented the number or combinations that result in 0 black AND 5 red cards.

Let H represent the number of hands with at most 1 black card:

$$H = {}_{26}C_1 \cdot {}_{26}C_4 + {}_{26}C_0 \cdot {}_{26}C_5$$

$$H = 26 \cdot 14\,950 + 1 \cdot 65\,780$$

$$H = 454\,480$$

Since the condition "at most 1 black card" is met by case 1 OR case 2, I knew I should add the combinations for each case.

There are 454 480 different five-card hands that contain at most one black card.

Your Turn

Solve the problem above using indirect reasoning. To do this, you will need to consider the total number of five-card hands as well as the number of five-card hands that do not meet the condition "at most one black card."

In Summary

Key Idea

- When solving counting problems, you need to determine if order plays a role in the situation. Once this is established, you can use the appropriate permutation or combination formula.

Need to Know

- Once you have established whether a problem involves permutations or combinations you can also use these strategies:
  - Look for conditions. Consider these first as you develop your solution.
  - If there is a repetition of r of the n objects to be eliminated, it is usually done by dividing by $r!$.
  - If a problem involves multiple tasks that are connected by the word AND, then the Fundamental Counting Principle can be applied: multiply the number of ways that each task can occur.
  - If a problem involves multiple tasks that are connected by the word OR, the Fundamental Counting Principle does not apply: add the number of ways that each task can occur. This typically is found in counting problems that involve several cases.

CHECK Your Understanding

1. Identify whether each situation involves permutations or combinations. Explain how you know.

a) Choose 3 toppings for a pizza from 25 different possibilities.

b) Choose a CEO, president, and vice-president from a group of 20 candidates.

c) Determine the number of outcomes possible when rolling 3 dice: 1 red, 1 blue, and 1 white.

d) Determine the number of ways 5 children from a group of 11 can start in a game of pickup basketball.

2. Consider two situations:

- Situation A: A committee of 3 is to be selected from a group of 10 people.
- Situation B: An executive committee consisting of a president, a vice-president, and a secretary is to be selected from 10 people.

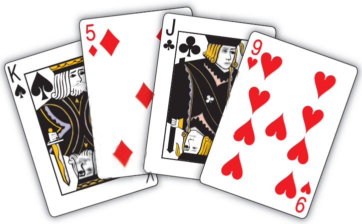
Determine which of these situations involves combinations and which involves permutations. Explain your answer.

3. Maddy arrived at an auction, and there were only 8 items left to bid on. She likes all 8 items but especially likes 3 items. She can afford to have winning bids on only 3 items. How many ways can she bid on 3 items under each of the following conditions?

a) She bids on only her 3 favourite items.

b) She bids on any 3 of the 8 items.

PRACTISING



4. From a standard deck of 52 cards, how many different four-card hands are there with one card from each suit?

5. How many ways can the top five cash prizes be awarded in a lottery that sold 200 tickets

a) if each ticket is not replaced when drawn?

b) if each ticket is replaced when drawn?

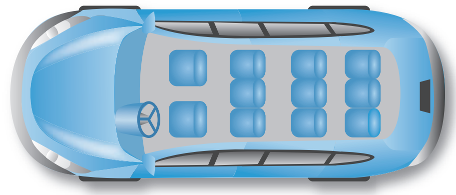
6. How many ways can the 5 starting positions on a basketball team (1 centre, 2 forwards, and 2 guards) be filled from a team of 2 centres, 4 guards, and 6 forwards?

7. How many ways can five different pairs of identical teddy bears be arranged in two rows of five for a photograph?

8. Five different signal flags fly on the flag pole of a coast guard ship. You can send signals using one or more of the flags. How many different signals can be sent using at least three of these flags?

9. Six different types of boats have pulled into a marina and want to dock at the six available slips. The six slips are adjacent to each other. How many ways can the six boats dock so that the two cabin cruisers, which are travelling together, are docked next to each other?

10. Mark is the coach of a boys' basketball team. He has rented a passenger van as shown to drive to a weekend tournament. How many ways can the 10 players sit in the van if Mark drives and there must be three players on each bench? State any assumptions you are making.



11. **a)** How many different arrangements are possible for the letters in the word FUNNY if there are no conditions?

b) How many different arrangements are possible if each arrangement must start and end with an N?

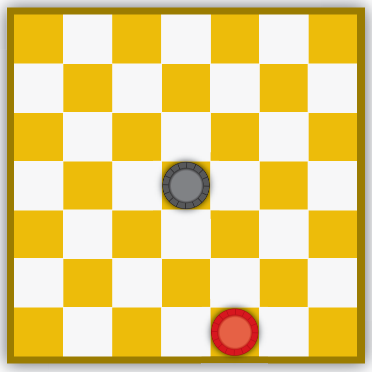
12. A combination lock opens when the right sequence of three numbers from 0 to 99 is used. The same number may be used more than once. How many sequences are there that consist entirely of odd numbers?

13. How many different routes can you take to get to a location five blocks south and six blocks east, if you travel only south or east?

14. Three vehicles are taking 16 musicians to a concert: a 5-person car driven by Joe, a 4-person car driven by Kuami, and a 7-passenger van driven by Angela. How many ways can the 16 people be assigned to the 3 vehicles?

15. On the game board shown, the red checker is allowed to move toward the top of the board diagonally left or right. If the black checker is encountered, the red checker cannot move into its square or jump over it. Determine the number of paths the red checker can follow from its starting position to any yellow square along the top of the board.

top of board



start

16. How many different five-card hands that contain at most three hearts can be dealt from a standard deck of playing cards?

Closing

17. Create a flow chart that you could use to help you make decisions about how to solve counting problems.

Extending

18. This week, six boys and seven girls signed up for a ski trip. Only six students can go, so they are to be selected at random. What is the probability that there will be three boys and three girls on the trip?

19. How many four-letter arrangements can be made using the letters in the word ALASKA?

20. How many three-letter arrangements can be made using the letters in the word BOOKS?

History | Connection

Computer Codes

ASCII is an acronym for American Standard Code for Information Interchange. It is a code for representing English characters that appear on computer keyboards.

The ASCII code uses the numbers from 0 to 127. For example, the ASCII code for M is 77. Most computers use ASCII codes to represent text. This makes it possible to transfer text from one computer to another.

All the characters used in email messages are ASCII characters and so are all the characters used in HTML documents. Web browsers read the ASCII characters between carets, < and >, to interpret how to format and display HTML documents.

The standard ASCII codes from 0 to 31 are used to represent control codes for things like Tab, Shift, Esc, and Space Bar. The numbers 32 to 127 are used to represent upper- and lower-case letters, the digits 0 to 9, and other symbols such as #, $, and ?.

Computers communicate with each other using a binary code system. In this system, there are only two digits (0 and 1), which are represented by two states: off and on. ASCII codes are converted to binary code as a string of 0s and 1s.

A. Does order matter in the ASCII code system? Explain.

B. Do you think order matters in the binary code system? Explain.

C. How long would a string of 0s and 1s in the binary system have to be to represent all of the 128 ASCII codes? Explain how you know.