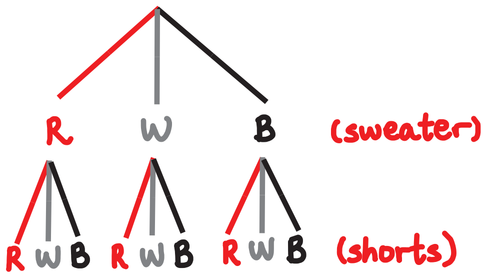
4.1 - COUNTING PRINCIPLES

Goal: Determine the Fundamental Counting Principle and use it to solve problems.

Example 1:

Hannah plays on her school soccer team. The soccer uniform has: three different sweaters: red, white, and black, and three different shorts: red, white, and black. How many different variations of the soccer uniform can the coach choose from for each game? Make a tree diagram.



9 uniforms
$(3 \times 3 = 9)$

Try:

A toy manufacturer makes a wooden toy in three parts. Determine how many different coloured toys can be produced? Make a tree diagram.

Part 1: The top part may be coloured red or blue 2

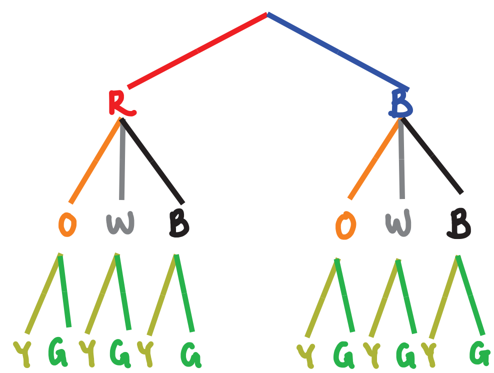
Part 2: The middle part may be orange, white, or black 3

Part 3: The bottom part may be yellow or green 2

middle

$2 \times 3 \times 2 = 12$ different toys

top bottom



Fundamental Counting Principle
If there are *a* ways to perform one task and *b* ways to perform another, then there are $a \times b$ ways of performing both.

Consider a task made up of several stages. The fundamental counting principle states that if the number of choices for the first stage is a , the number of choices for the second stage is b , the number of choices for the third stage is c , etc.. then the number of ways in which the task can be completed is $a \times b \times c \times ....$

Solve a Counting Problem by Extending the Fundamental Counting Principle

Example 2:

A luggage lock opens with the correct three-digit code. Each wheel rotates through the digits 0 to 9.

10 digits

a. How many different three-digit codes are possible?

$10 \times 10 \times 10 = 1000$ codes

b. Suppose each digit can be used only once in a code. How many different codes are possible when repetition is not allowed?

$10 \times 9 \times 8 = 720$ codes

Try:

A vehicle license plate consists of 3 letters followed by 3 digits. How many license plates are possible if:

a. there are no restrictions on the letters or digits used?

$26 \times 26 \times 26 \times 10 \times 10 \times 10 = 26^3 \times 10^3 = 17\,576\,000$ license plates

b. no letter may be repeated?

$26 \times 25 \times 24 \times 10 \times 10 \times 10 = 15\,600\,000$ license plates

8 choices left

c. the first digit cannot be zero and no digits can be repeated?

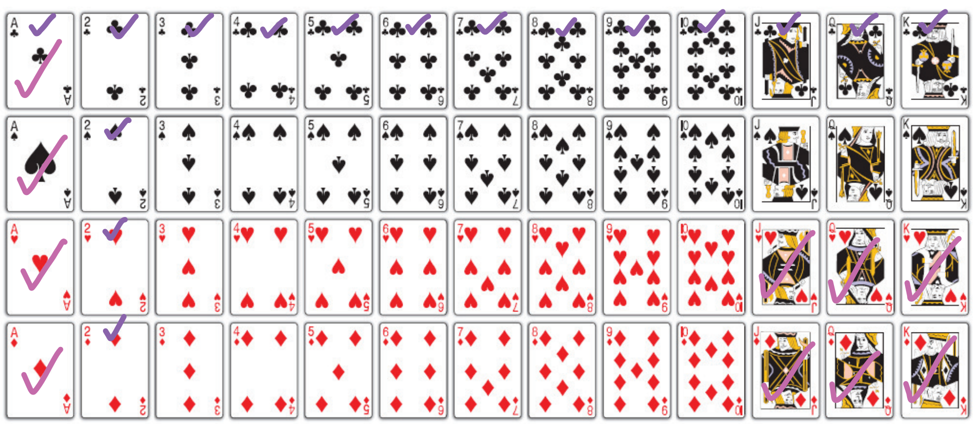
$26 \times 26 \times 26 \times 9 \times 9 \times 8 = 11\,389\,248$ license plate

1 less but zero is "back"

not zero

Example 3: Solving a counting problem when the Fundamental Counting Principle does not apply

A standard deck of cards contains 52 cards as shown.



Count the number of possibilities of drawing a single card and getting:

a. either a red face card or an ace

10 cards

b. either a club or a two

16 cards

Assignment: p. 235 #4 – 6, 9 – 11, 13, 15

4.2 - INTRODUCING PERMUTATIONS AND FACTORIAL NOTATION

Permutation is an arrangement of distinguishable objects in a definite order. For example, the objects *a* and *b* have two permutations, *ab* and *ba*.

Solve a Counting Problem Where Order Matters

Example 1:

Determine the number of arrangements that 4 children can form while lining up to go to the washroom.

Alphonse, Beatrice, Clarence, Deborah

$4 \times 3 \times 2 \times 1 = 24$ arrangement

Example 2:

When you press the "shuffle" button on an i-Pod, it plays a list of the songs (all songs will be played only once). If the i-Pod has 6 songs on it, how many playlists of the songs are possible?

$6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ playlists

Try:

How many ways can 5 different books, Math, Chemistry, Physics, English and Biology be arranged on a shelf?

$5 \times 4 \times 3 \times 2 \times 1 = 5! = 120$ ways

Factorial Notation!

$1! = 1$

$2! = 2 \times 1 = 2$

$3! = 3 \times 2 \times 1 = 6$

$4! = 24$

$5! = 120$

$6! = 720$

$0! = 1$

A concise representation of the product of consecutive descending natural numbers:

$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$

$(n+1)! = (n+1) \times n \times (n-1) \times \cdots \times 3 \times 2 \times 1$

$(n-1)! = (n-1) \times (n-2) \times (n-3) \times \cdots \times 3 \times 2 \times 1$

In the expression $n!$, the variable n is defined only for values that belong to the set of whole numbers; that is, $n \in \{0, 1, 2, 3, ...\}$. Please note that 0! is defined to be 1.

Evaluate Numerical Expressions Involving Factorial Notation

Example 3: Evaluate the following.

$10! = 3,628,800$

$$\frac{12!}{9!\,3!} = \frac{12\times 11\times 10\times \cancel{9!}}{\cancel{9!}\times 3!}$$

$$= \frac{\overset{2}{\cancel{12}}\times 11\times 10}{\underset{1}{\cancel{6}}}$$

$$= 220$$

Simplify an Algebraic Expression Involving Factorial Notation

Example 4: Simplify each expression, where $n \in N$.

$(n+3)(n+2)!$

$= (n+3)!$

$$\frac{(n+1)!}{(n-1)!} = \frac{(n+1)n\cancel{(n-1)!}}{\cancel{(n-1)!}} = (n+1)n$$

Example 5: Write each expression without using the factorial symbol.

$$\frac{(n+2)!}{n!} = \frac{(n+2)(n+1)\cancel{n!}}{\cancel{n!}}$$

$$= (n+2)(n+1)$$

$$\frac{(n-3)!}{n!} = \frac{\cancel{(n-3)!}}{n(n-1)(n-2)\cancel{(n-3)!}}$$

$$= \frac{1}{n(n-1)(n-2)}$$

Try: Calculate the value of:

$$\frac{43!}{40!} = \frac{43\cdot 42\cdot 41\cdot \cancel{40!}}{\cancel{40!}} = 43\cdot 42\cdot 41$$

$$= 74,046$$

$$\frac{37!}{33!\,4!} = \frac{37\cdot 36\cdot 35\cdot 34\cdot \cancel{33!}}{\cancel{33!}\,4!}$$

$$= \frac{1585080}{24}$$

$$= 66,045$$

Solve an Equation Involving Factorial Notation

Example 5: Solve for n.

$\frac{n!}{(n-2)!} = 90$, where $n \in I$

$\frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}} = 90$

$n(n-1) = 90$

$n^2 - n = 90$

$n^2 - n - 90 = 0$

$(n-10)(n+9) = 0$

$n = 10$ or $\cancel{-9}$

n is positive

$\frac{\overset{1}{\cancel{3}}(n+1)!}{(n-1)!} = \overset{42}{\cancel{126}}$, where $n \in I$

$\frac{(n+1)!}{(n-1)!} = 42$

$\frac{(n+1)n\cancel{(n-1)!}}{\cancel{(n-1)!}} = 42$

$(n+1)n = 42$

$7 \cdot 6 = 42$

$\therefore n = 6$

Assignment: p. 244 #5 – 8, 11 – 13

4.3 - PERMUTATIONS WHEN ALL OBJECTS ARE DISTINGUISHABLE

Permutation is an arrangement of distinguishable objects in a definite order. For example, the objects *a* and *b* have two permutations, *ab* and *ba*.

${}_nP_r = \dfrac{n!}{(n-r)!}$ is the notation commonly used to represent the number of permutations that can be made from a set of n different objects where only r of them are used in each arrangement, and $0 \le r \le n$.

Example 1: Solving a permutation problem where only some of the objects are used in each arrangement

Luke has downloaded 10 new songs from an online music store. He wants to create a playlist using 6 of these song arranged in any order. How many different 6-song playlist can be created from his new downloaded songs?

10 songs $\times 9 \times 8 \times 7 \times 6 \times 5 = 151{,}200$ playlists

${}_{10}P_6 = 10!/(10-6)! = 10!/4! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times \cancel{4!}/\cancel{4!} = 151{,}200$ playlists

Example 2 Defining 0!

Use the formula for ${}_nP_r$ to show that $0! = 1$.

${}_nP_n = n!/(n-n)! = n!/0! = n! \leftarrow$ (logically, see 4.2)

$\therefore 0! = 1$

Example 3: Solving a permutation problem involving cases

An online store allows its users to register with passwords with a minimum of 5 characters and a maximum of 7 characters. The passwords can use any digits from 0 to 9 and/or any letters of the alphabet. The passwords are case sensitive. (Each character can be used once)

How many different passwords are possible?

uppercase and lowercase

10 digits + 2 × (26 letters) = 62 characters

${}_{62}P_5 = 776{,}520{,}240$

+

${}_{62}P_6 = 44{,}261{,}653{,}680$

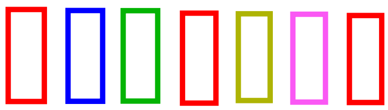
+

${}_{62}P_7 = 2{,}478{,}652{,}606{,}080 = 2{,}523{,}690{,}780{,}000$

Example 4: Solving a permutation problem with conditions

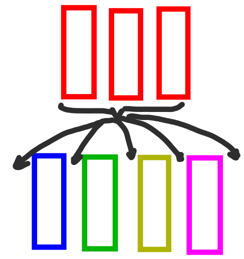
At a used car lot, seven different car models are to be parked close to the street for easy viewing.

a. The three red cars must be parked so that there is a red car at each end and the third red car is exactly in the middle. How many ways can the seven cars be parked?
b. The three red cars must be parked side by side. How many ways can the seven cars be parked?



$_3P_3 = 3! = 6$

$_4P_4 = 4! = 24$ $\quad 6 \times 24 = 144$ arrangement



the red cars are grouped together and treated as one object.

$_3P_3 = 3! = 6$ ← arrange red cars

$_5P_5 = 5! = 120$ ← non-red + group

$6 \times 120 = 720$ arrangements

Example 5: Comparing arrangements created with and without repetition

A social insurance number (SIN) in Canada consists of a nine-digit number that uses the digits 0 to 9. Compare the number of possible social insurance numbers with and without repetitions.

$10^9 = 1000\,000\,000$ SIN's

$_{10}P_9 = \frac{10!}{(10-9)!} = \frac{10!}{1!} = 10!$

$= 3{,}628{,}800$ SIN's

Assignment: p. 255 #5, 6, 8 – 11, 13, 15 – 17

4.4 - PERMUTATION WHEN OBJECTS ARE IDENTICAL

Goal: Determine the number of permutations when some objects are identical.

Example 1:

Three ~~can~~ cans are to be put on a shelf.

C

P

R

a. List all permutations.

CPR CRP
PCR PRC
RCP RPC

3 possible cans for 1st spot

$3 \times 2 \times 1 = 6$ permutations $= 3!$

last one

2 remaining possibilities

b. If the Red Bull is replaced by another Coca-Cola, list all permutations.

CPC ~~CCP~~
PCC ~~PCC~~
CCP ~~CPC~~

only 3 permutations

3 cans

$$\frac{3!}{2!} = 6/2 = 3 \text{ permutations}$$

2 Coca-Cola

> The number of permutations of n objects, where *a* are identical, another *b* are identical, another *c* are identical, and so on, is $\frac{n!}{a!b!c!\ldots}$.

Example 2:

Beck bought a carton containing 6 mini boxes of cereal. There are 3 boxes of Cheerios, 2 boxes of Fruit Loops, and 1 box of Mini-Wheats. Over a six day period, Beck plans to eat the contents of one box of cereal each morning. How many different orders are possible?

6 boxes

$$\frac{6!}{3!\,2!}$$

Fruit Loops

Cheerios

$= 720/(6 \cdot 2) = 60$ permutations

Try:

Naval signals are made by arranging coloured flags in a vertical line and the flags are then read from top to bottom. How many signals using six flags can be made if you have:

a. 3 red, 1 green, and ~~3~~ 2 blue flags

$$\frac{6!}{3!2!} = 60$$

b. 2 red, 2 green, and 2 blue flags

$$\frac{6!}{2!2!2!} = 90$$

Example 3:

Determine the number of permutations of all the letters in each of the following words.

a. OGOPOGO

7 letters

$$\frac{7!}{4!\,2!} = \frac{7\cdot\cancel{6}^{3}\cdot 5\cdot\cancel{4!}}{\cancel{4!}\cdot\cancel{2!}}$$

4 O's 2 G's

$= 105$

b. STATISTICIAN

$$\frac{12!}{2!\,3!\,2!\,3!} = 3,326,400$$

S T A I

Solve a Conditional Permutation Problem Involving Identical Objects

Example 4:

How many ways can the letters of word CANADA be arranged, if the first letter must be N and the last letter must be C?

N _ _ _ _ C

3A's, 1D

$4!/3! = 4$ permutations

Try:

Tina is playing with a tub of building blocks. The tub contains 3 red blocks, 5 blue blocks, 2 yellow blocks, and 4 green blocks. How many different ways can Tina stack the blocks in a single tower, if there must be a yellow block at the bottom of the tower and a yellow block at the top.

Y _ _ _ _ _ _ _ _ _ _ _ _ Y

3 red

5 blue

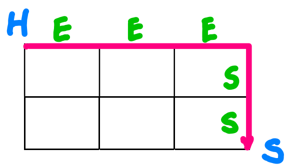
4 green

$12!/3!5!4! = 27,720$ permutations

Solve a Permutation Problem Involving Routes

Example 5:

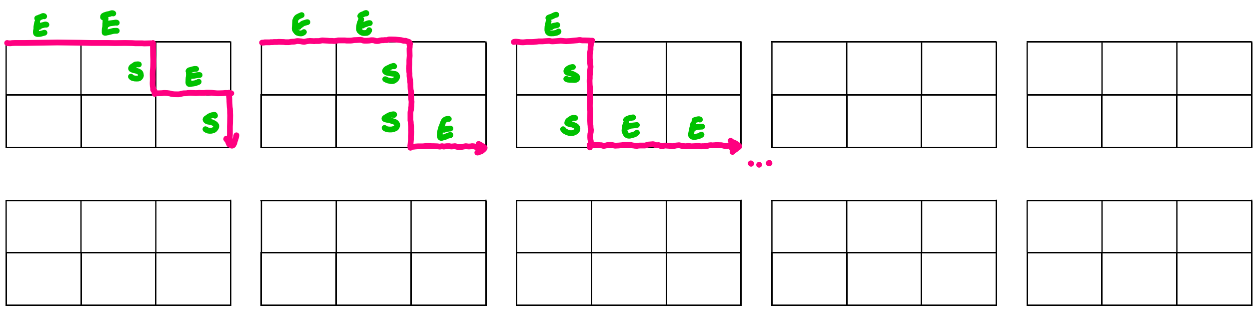
Julie's home is two blocks north and three blocks west of her school. How many routes can Julie take from home to school if she always travels either south or east?

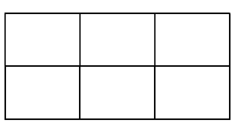


5 blocks

$$\frac{5!}{3!\,2!} = 10 \text{ routes}$$

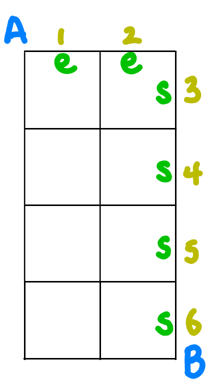
3 east 2 south





Try:

On the following grid, how many different paths can A take to B, assuming one can only travel east and south? Explain.



$$\frac{6!}{2!\,4!} = 15$$

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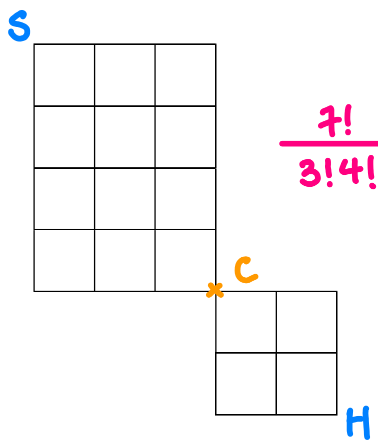
Example 6:

A supervisor of the city bus department is determining how many routes there are from the bus station to the concert hall. Determine the number of routes possible if the bus must always move closer to the concert hall.



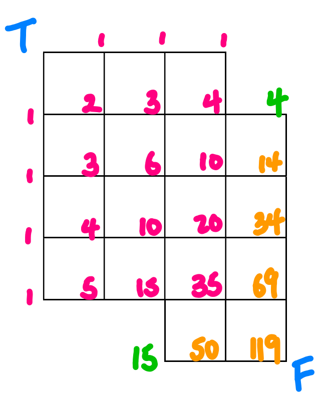
$$\frac{7!}{3!4!} = 35$$

$$35 \times 6 = 210 \text{ routes}$$

$$\frac{4!}{2!2!} = 6$$

Example 7:

A taxi company is trying to find the quickest route during rush hour traffic from the train station to the football stadium. How many different routes must be considered if at each intersection the taxi must always move closer to the football stadium?



119 routes

Assignment: p. 267 #4 – 11, 13

4.5 & 4.6 - EXPLORING COMBINATIONS

Goal: Solve problems involving combinations.

Example 1: Calculating Combinations

If 5 sprinters compete in a race final, how many different ways can the medals for *first*, *second*, and *third* place be awarded?

${}_nP_r = {}_5P_3 = 5!/(5-3)! = 5!/2! = 60$ ways

If 5 sprinters compete in a qualifying heat of a race, how many different ways can the sprinters qualify?

qualifies

QQQDD

does not qualify

$5!/3!2! = 5\cdot4\cdot\cancel{3!}/\cancel{3!}\cdot 2! = 20/2 = 10$ ways

${}_nC_r = {}_5C_3 = 5!/3!(5-3)! = 10$ ways

A permutation is an **arrangement** of elements in which the order of the arrangement is taken into account. A combination is a **selection** of element in which the order of selection is NOT taken into account.

Example 2: Solving a Simple Combination Problem

Three students from a class of 10 are to be chosen to go on a school trip. In how many ways can they be selected?

${}_{10}C_3 = 10!/3!(10-3)! = 10!/3!7! = 120$ ways

SSSNNNNNNN

Combination of "*n*" different objects taken "*r*" at a time is:

$$ {}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!} $$

There are 16 students in a class. Determine the number of ways in which four students can be chosen to complete a survey.

${}_{16}C_4 = 16!/4!(16-4)! = 16!/4!12! = 1820$ ways

Example 3: Solving a Combination Problem Using the Fundamental Counting Principle

The Athletic Council decides to form a sub-committee of 7 council members to look at how funds raised should be spent on sport activities in the school. There are a total of 15 athletic council members, 9 males and 6 females. The sub-committee must consist of exactly 3 females. Determine the number of ways of selecting the sub-committee.

$7 - 3 = 4$ spots left

male

$\underset{\text{female}}{{}_6C_3} \times {}_9C_4 = 20 \times 126 = 2520$ possible sub-committee

A basketball coach has 5 guards and 7 forwards on his basketball team. In how many different ways can he select a starting team of two guards and three forwards?

$\underset{\text{guards}}{{}_5C_2} \times \underset{\text{forwards}}{{}_7C_3} = 10 \times 35 = 350$ possible starting lineups

Example 4: Solving a Combination Problem by Considering Cases

opposite: having no teachers

A planning committee is to be formed for a school-wide Earth Day program. There are 13 volunteers: 8 teachers and 5 students. How many ways can the principal choose a 4-person committee that has at least 1 teacher?

$\underset{\text{1 teacher}}{{}_8C_1 \times {}_5C_3} + \underset{\text{2 teachers}}{{}_8C_2 \times {}_5C_2} + \underset{\text{3 teachers}}{{}_8C_3 \times {}_5C_1} + \underset{\text{4 teachers}}{{}_8C_4 \times {}_5C_0}$

$80 + 280 + 280 + 70 = 710$ ways

$\underset{\text{no restrictions}}{{}_{13}C_4} - \overset{\text{no teachers}}{{}_8C_0 \times {}_5C_4} = 715 - 5 = 710$ ways

must have at least 1 teacher

An all-night showing at a movie theatre is to consist of five movies. There are fourteen different movies available, ten disaster movies and four horror movies. How many possible schedules include at least four disaster movies?

${}_{10}C_4 \times {}_4C_1 + {}_{10}C_5 \times {}_4C_0$

$= 840 + 252 = 1092$ possible schedules

Assignment: p. 280 #5 – 13

4.7 – SOLVING COUNTING PROBLEMS

Goal: Solve counting problems that involve permutations and combinations.

When solving counting problems, you need to determine if order plays a role in the situation. Once this is established, you can use the appropriate permutation or combination formula.

Solve a Permutation Problem with Conditions

Example 1:

A piano teacher and her students are having a group photograph taken. There are three boys and five girls. The photographer wants the boys to sit together and the girls to sit together for one of the poses. How many ways can the students and teacher sit in a row of nine chairs for this pose?

T BBB GGGGG $3! = 6$ arrangements $6 \times 1 \times 6 \times 120 = 4320$ ways

BBB T GGGGG $1! = 1$ (teacher)

⋮

$3! = 6$ (boys) $5! = 120$ (girls)

Example 2:

In how many ways can all of the letters of the word ORANGES be arranged if…

a. There are no restrictions?

$7! = 5040$ ways

b. The first term must be an N?

N _ _ _ _ _ _ $1 \times 6! = 720$ ways

c. The vowels must be together in the order O, A, and E?

$\underset{1}{\underline{(O,A,E)}}$ $\underset{2}{\underline{\ }}\ \underset{3}{\underline{\ }}\ \underset{4}{\underline{\ }}\ \underset{5}{\underline{\ }}$ 5 objects to arrange $5! = 120$ ways

Try:

In how many of the arrangements of the letters of the word BRAINS are the vowels together?

$2! = 2$ vowels $5! = 120$ 4 consonants + 1 group of vowels

$2 \times 120 = 240$ arrangements

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Solve a Combination Problem Involving Multiple Choices

Example 3:

A standard deck of 52 playing cards consists of 4 suits (spades, hearts, diamonds, and clubs) of 13 cards each.

a. How many different 5-card hands can be formed?

$_{52}C_5 = 52!/5!(52-5)! = 2,598,960$ hands

b. How many different 5-card hands can be formed that consist of all hearts?

$_{13}C_5 = 13!/5!(13-5)! = 1,287$ hands

c. How many different 5-card hands can be formed that consist of all face cards?

$_{12}C_5 = 12!/5!(12-5)! = 792$ hands

d. How many different 5-card hands can be formed that consist of 3 hearts and 2 spades?

$_{13}C_3 \times {_{13}C_2} = 286 \times 78 = 22\,308$ hands

e. How many different 5-card hands can be formed that consist of exactly 3 hearts?

not hearts

$_{13}C_3 \times {_{39}C_2} = 286 \times 741 = 211,926$ hands

hearts

Try:

A group of 4 journalists is to be chosen to cover a murder trial. There are 5 male and 7 female journalists available. How many possible groups can be formed consisting of exactly 2 men and 2 women?

$_5C_2 \times {_7C_2} = 10 \times 21 = 210$ possible groups

Solve a Combination Problem Involving Cases

Example 4:

The Student Council decides to form a sub-committee of five council members to look at how funds raised should be spent on the students of the school. There are a total of 11 student council members, 5 males and 6 females. How many different ways can the sub-committee consist of at least three females?

$_5C_2 \times {}_6C_3 + {}_5C_1 \times {}_6C_4 + {}_5C_0 \times {}_6C_5$

$= 10 \times 20 + 5 \times 15 + 1 \times 6 = 200 + 75 + 6 = 281$ ways

Example 5:

Consider a standard deck of 52 cards. How many different five card hands can be formed containing

a. At least 1 red card?

opposite: no red cards

no red cards

$_{52}C_5 - {}_{26}C_5 = 2\,533\,180$ hands

all possibilities   at least one red.

b. At most 2 kings?

$\underbrace{{}_4C_0 \times {}_{48}C_5}_{\text{no kings}} + \underbrace{{}_4C_1 \times {}_{48}C_4}_{\text{1 king}} + \underbrace{{}_4C_2 \times {}_{48}C_3}_{\text{2 kings}}$

$= 1\,712\,304 + 778\,320 + 103\,776 = 2\,594\,400$ hands

Try.

The Athletic Council decides to form a sub-committee of 6 council members to look at a new sports program. There are a total of 15 student council members, 6 females and 9 males. How many different ways can the sub-committee consist of at most one male? 55 ways, but how?

Assignment: p. 288 #3 – 8, 10 – 14, 16