
SALE

Chapter

1

Financial Mathematics: Investing Money

LEARNING GOALS

You will be able to develop your number sense in financial applications by

- Understanding and comparing the effects of simple interest and compound interest
- Determining how changes in the variables of an investment affect the return
- Being aware of a variety of different investment instruments
- Comparing different investment strategies

? What do you think it means to be financially literate, and how will being financially literate help you achieve your goals?

1 Getting Started

Investing to Travel Western Canada

Deb and her best friend Kali have always talked about taking a road trip through Western Canada. They want to go in July so they can attend the Calgary Stampede, Edmonton's Capital EX, the Winnipeg Folk Festival, the Craven Country Jamboree in Saskatchewan, and the Williams Lake Stampede in British Columbia. Deb has a goal of saving $2250 to pay for her portion of the gas, hotels, tickets, and food. She has $200 in a savings account that earns 0.25% interest each month on the minimum monthly balance. At the end of each month, she makes a deposit of $100.

? How long will it take Deb to meet her goal of saving $2250?

The Craven Country Jamboree, held every year in Craven, in the Qu'Appelle Valley of Saskatchewan, draws visitors from across North America.

A. Predict how long it will take for Deb to meet her goal. Explain your prediction.

B. How much interest will she earn during the first month?

C. At the end of the first month, how much money will Deb have in her savings account?

D. Copy and complete the table below to determine how long it will take Deb to meet her goal. Compare your results with your prediction.

Month	Balance at Start of Month ($)	Interest Earned ($)	Deposits ($)	Balance at End of Month ($)
1	200.00		100.00	
2				
3				

WHAT DO You Think?

Decide whether you agree or disagree with each statement. Explain your decision.

1. Interest earned on an investment over time is a linear relationship.
2. Interest per annum means that interest is calculated and earned yearly.
3. An investment of $1000 at 5% annual interest for 10 years yields the same result as an investment of $1000 at 10% annual interest for 5 years.
4. A retirement investment portfolio, involving regular deposits, started when you are 25 years old is better than a retirement investment portfolio started when you are 45 years old.
5. The most important factor when building an investment portfolio is the rate of return.

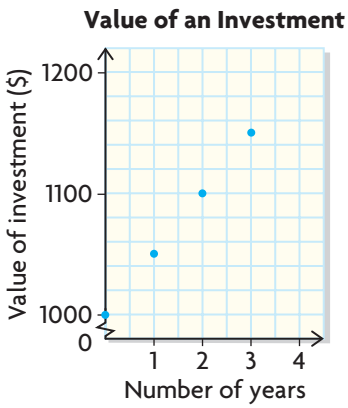
1.1 Simple Interest

YOU WILL NEED

- calculator
- graph paper
- straightedge

EXPLORE...

- An amount of money was invested. Interpret the graph below to determine
  a) how much money was invested,
  b) the value of the investment at the end of the first year, and
  c) the rate at which the value grew between the beginning and end of the first year.



term
The contracted duration of an investment or loan.

interest
The amount of money earned on an investment or paid on a loan.

fixed interest rate
An interest rate that is guaranteed not to change during the term of an investment or loan.

principal
The original amount of money invested or loaned.

GOAL

Solve problems that involve simple interest.

INVESTIGATE the Math

Sera is 20 years old and needs money to pay for college. When she was born, her grandparents bought her a $500 Canada Savings Bond (CSB) with a **term** of 10 years. They chose a CSB as an investment because they liked the security of loaning money to the government. The **interest** earned was determined using a **fixed interest rate** of 6% per year on the original investment and was paid at the end of each year until Sera's 10th birthday.

? How can you determine the current value of Sera's CSB?

A. How much interest was earned on the **principal** by the end of the first year?

B. Determine the **simple interest** earned each year, the accumulated interest, and the value of the investment for the first 4 years. Organize your calculations in a table like the one below.

simple interest

The amount of interest earned on an investment or paid on a loan based on the original amount (the principal) and the simple interest rate.

Year	Value of Investment at Start of Year ($)	Simple Interest Earned Each Year ($)	Accumulated Interest ($)	Value of Investment at End of Year ($)
0				500
1	500			
2				
3				
4				

C. Is the simple interest earned each year constant or variable? Explain.

D. Describe the relationship between the number of years, the interest earned each year, and the accumulated interest.

E. Use the relationship from part D to predict the value of the investment after 10 years.

F. Graph the growth of the investment until its **maturity** at 10 years using "Time (years)" as the domain and "Value of the investment ($)" as the range. Is your prediction in part E supported by your graph?

maturity

The contracted end date of an investment or loan, at the end of the term.

Reflecting

G. Describe your graph. What does the shape of your graph tell you about the type of growth?

H. What do the y-intercept and slope represent for the investment?

I. Describe how the value of the investment would change at maturity in each situation below, compared with the original situation. Explain why.

a) If the principal was $1000, but the interest was still 6% for 1 term of 10 years

b) If the principal was $500, but the interest was 5% for 1 term of 10 years

c) If the principal was $500 and the interest was 6%, but for 2 terms of 10 years

Communication *Tip*

Interest rates are communicated as a percent for a time period. Since most often the time period is per year or **per annum** (abbreviated as **/a**), a given percent is assumed to be annual unless otherwise stated. For example, an interest rate of 4% means 4%/a or 4% interest per year.

APPLY the Math

EXAMPLE 1 Solving a simple interest problem

Marty invested in a $2500 guaranteed investment certificate (GIC) at 2.5% simple interest, paid annually, with a term of 10 years.

a) How much interest will accumulate over the term of Marty's investment?

b) What is the **future value** of his investment at maturity?

c) Use Marty's investment to write an algebraic expression that could be used to determine the future value of any investment earning simple interest.

future value
The amount, A, that an investment will be worth after a specified period of time.

Marty's Solution

a) Amount of interest that will accumulate in 1 year:
$(0.025)(2500) = 62.5$

I multiplied the interest rate (as a decimal) by the principal.

In 1 year, $62.50 in interest will accumulate.

Since simple interest earned is a constant amount, and the simple interest earned in 1 year is $62.50, 10 times that amount will accumulate in 10 years.

$(62.5)(10) = 625$

In 10 years, $625 in interest will accumulate.

I noticed that the accumulated interest was the product of the principal, the interest rate, and the number of years.

$i = Prt$

b) Principal: $2500
Accumulated interest: $625

I knew that the future value at maturity is the sum of the principal and the accumulated interest, over the full term.

Future value of investment:
$2500 + $625 = $3125

c) Let A represent the amount, or future value of the investment,
P represent the principal amount invested,
t represent the time in years, and
r represent the interest rate per annum:

I wrote an equation to show that the future value or amount (A) is the sum of the principal (P) and the accumulated interest (Prt).

$A = P + Prt$
$A = P(1 + rt)$

I factored the expression on the right. I can use either of the equivalent forms of the equation.

Your Turn

How would an interest rate of 3.5% change the future value of Marty's investment?

EXAMPLE 2 Representing the growth of a simple interest investment

Sunni invested $15 000 in a savings account. Sunni earned a simple interest rate of 8%, paid semi-annually on her investment. She intends to hold the investment for 4.5 years, when she will withdraw all the money to buy a car. Determine the value of the investment at each half year until she withdraws the money.

Sunni's Solution: Using a table of values

Principal, P, is $15 000.
Rate, r, is 8%, or 0.08.
Time, t, is 0, 0.5, 1, 1.5, ..., 4.5 years.

Since interest is paid semi-annually, I knew that I could withdraw money after any half-year period and earn the interest up to that time.

Year	Value of Investment ($)
0	15 000
0.5	15 000 + 600 = 15 600
1.0	15 000 + 1200 = 16 200
1.5	15 000 + 1800 = 16 800
2.0	15 000 + 2400 = 17 400
2.5	18 000
3.0	18 600
3.5	19 200
4.0	19 800
4.5	20 400

I set up a table with a row for each half year. I determined the value of the investment at the end of each half year by adding the principal and the accumulated interest ($P + Prt$). For the first half year, this was 15 000 + 15 000(0.08)(0.5) or 15 000 + 600.

After calculating the value up to 2 years, I noticed a pattern: the investment grew by a constant amount, $600 every half year. I used the pattern to determine the value for each half-year interval until 4.5 years.

$A = P + Prt$

$A = 15\,000 + (15\,000)(0.08)(4.5)$
$A = 20\,400$

I verified the value at 4.5 years using the simple interest formula.

Taj's Solution: Using a graph

$A = P + Prt$

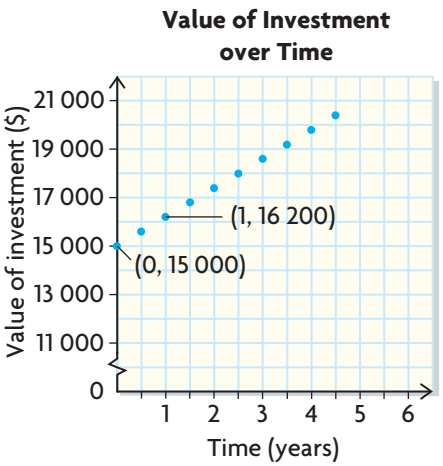
$A = 15\,000 + (15\,000)(0.08)t$
$A = 15\,000 + 1200t$
$A = 1200t + 15\,000$

I substituted the given information into the formula for future value of a simple interest investment.

The A-intercept is 15 000.
The slope is 1200.

I noticed that the equation of the line was in the form $y = mx + b$.

I knew that b or P was the A-intercept ($15 000) and m or Pr was the slope (1200). I also knew that I needed only two points to graph a linear relation.



Because interest is earned only at the end of each half-year period, t is not continuous in this situation, so the graph had to be a series of discrete points.

I created a grid and plotted the principal at (0, 15 000). I used the slope to determine a second point, (1, 16 200). Since the slope $\left(\frac{\text{rise}}{\text{run}}\right)$ is 1200, there is a rise of $1200 for every run of 1 year.

I used my ruler to extrapolate the remaining eight points, one at each half-year interval.

The value of the investment will be $20 400 after 4.5 years.

Your Turn

How did the formula for the future value of a simple interest investment help Sunni create the table? How did it help Taj create the graph?

EXAMPLE 3 Determining the duration of a simple interest investment

Ingrid invested her summer earnings of $5000 at 8% simple interest, paid annually. She intends to use the money in a few years to take a holiday with a girlfriend.

a) How long will it take for the future value of the investment to grow to $8000?
b) What is Ingrid's **rate of return**?

rate of return
The ratio of money earned (or lost) on an investment relative to the amount of money invested, usually expressed as a decimal or a percent.

Ingrid's Solution

a) $A = P + Prt$

P is \$5000.
r is 8%, or 0.08.
A is \$8000.

$$8000 = 5000 + (5000)(0.08)t$$
$$3000 = 400t$$
$$7.5 = t$$

I knew P, r, and A. I determined t by substituting these known values into the formula $A = P + Prt$ and solving for t.

Because I needed to isolate t, I knew that the $A = P + Prt$ form of the equation would have fewer solution steps than the $A = P(1 + rt)$ form would.

It will take 8 years for the future value of the investment to be at least \$8000.

I knew 7.5 years would not work because the interest is paid annually. This meant that I had to round up to the next whole year. It also meant that, at 8 years, the future value would be more than \$8000.

b) After 8 years:

$A = P + Prt$
$A = 5000 + (5000)(0.08)(8)$
$A = 8200$

At 8 years, the future value will be \$8200.

Interest earned:
\$8200 − \$5000 = \$3200

I determined the interest earned by subtracting the principal from the future value.

$$\text{Rate of return} = \frac{3200}{5000}$$

$$\text{Rate of return} = 0.64$$

I compared the interest earned with the principal to determine the rate of return.

The rate of return is 64% over 8 years.

Your Turn

How would each situation below change the length of time needed for the future value of Ingrid's investment to grow to \$8000 under simple interest conditions?

a) If Ingrid invested principal of only \$4000 at 8%, paid quarterly
b) If Ingrid earned 4% on principal of \$5000
c) If 8% interest on principal of \$5000 was paid semi-annually

EXAMPLE 4 Determining the rate of interest on a simple interest investment

Grant invested $25 000 in a simple interest Canada Savings Bond (CSB) that paid interest annually.

a) If the future value of the CSB is $29 375 at the end of 5 years, what interest rate does the CSB earn?

b) Grant cashed in the bond after 4.5 years because a house he had been admiring came up for sale and he needed a down payment. How much money did he have for the down payment?

Cilla's Solution

a) $A = P + Prt$

A is $29 375.
P is $25 000.
t is 5 years.

$$29\,375 = 25\,000 + (25\,000)(r)(5)$$
$$4375 = 125\,000r$$
$$r = 0.035$$

The interest rate was 3.5%.

I chose the $A = P + Prt$ form of the equation because I needed to isolate r. I knew that this form of the equation would have fewer solution steps.

I determined r by substituting the known values into the formula and solving for r.

The annual interest rate was expressed as a decimal, so I changed it to a percent.

I knew that the interest rate was an annual rate.

b) $A = P(1 + rt)$

P is $25 000.
r is 0.035.
t is 4 years.

$$A = 25\,000(1 + (0.035)(4))$$
$$A = 28\,500$$

Grant had $28 500 for a down payment.

I used a value of 4, instead of 4.5, for t because the interest is paid annually. Since Grant cashed in the CSB after 4.5 years, he did not earn interest for the last half year.

Your Turn

How would the interest rate change in each situation? Explain.

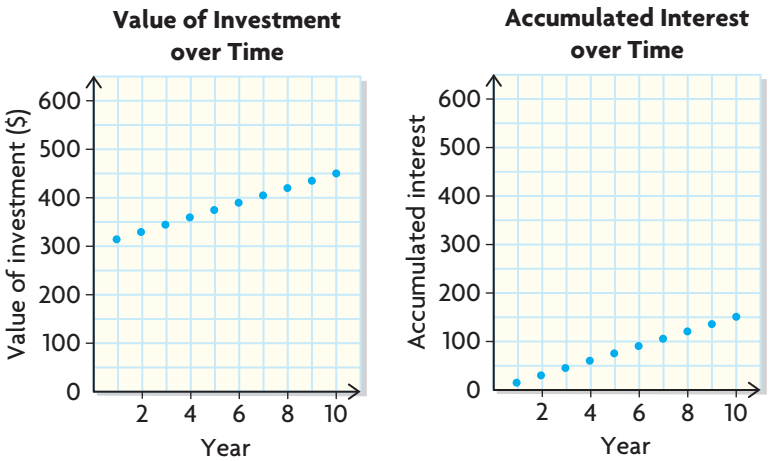
a) If Grant invested principal of $20 000 instead, and the CSB grew to $29 375 in 5 years

b) If it took 8 years for Grant's principal of $25 000 to grow to $29 375

In Summary

Key Ideas

- Simple interest is determined only on the principal of an investment.
- The value of an investment that earns simple interest over time is a linear function. The accumulated simple interest earned over time is also a linear function. Since the interest is paid at the end of each period, the growth is not continuous. For example, the following graphs show principal of $300 invested at 5% interest, paid annually, over a term of 10 years.



Need to Know

- The amount of simple interest earned on an investment can be determined using the formula

$$I = Prt$$

where I is the interest, P is the principal, r is the annual interest rate expressed as a decimal, and t is the time in years.

- The future value or amount, A, of an investment that earns simple interest can be determined using the formula

$$A = P + Prt$$
$$\text{or } A = P(1 + rt)$$

where P is the principal, r is the interest rate expressed as a decimal, and t is the time in years.

- Unless otherwise stated, an interest rate is assumed to be annual, or per annum.
- Even though interest rates are usually annual, interest can be paid out at different intervals, such as annually, semi-annually, monthly, weekly, and daily.

CHECK Your Understanding

1. Determine the future value of each investment if it earns simple interest.

a) 8-year term on a principal of $30 000 at 3.5%

b) 1.25% interest paid quarterly for 4 years on $10 000

c) 0.5% interest paid weekly for 2 years on $25 000

d) 20-year term at 7.4% on a deposit of $12 000

2. Cam has $5000 to invest. He wants his principal to grow to $6500 in 5 years so that he can afford a new drum kit.

a) What simple interest rate will allow him to meet his goal?

b) Suppose that interest is paid semi-annually and Cam withdraws all the money after 3.25 years. How much money will he have?

3. **a)** Principal of $1000 is invested at 5% simple interest, paid annually, for 5 years. What is the rate of return?

b) Which option below would yield the greatest future value? What is the rate of return for this option?

A. increasing the principal to $1050

B. increasing the interest rate to 6%

C. paying interest every 6 months

D. increasing the term to 6 years

PRACTISING

4. Char invested $4000 at a simple interest rate of 2.3%.

a) What is the value of her investment after 5 years?

b) What is its value after 10 years?

5. Both Brad and Chris purchased a $15 000 GIC.

- Brad's GIC has a term of 6 years and a simple interest rate of 3.2%.
- Chris's GIC has a term of 5 years at a simple interest rate of 3.3%.

Whose GIC will have the greater future value at maturity? Explain.

6. a) A $12 000 Canada Savings Bond has a term of 10 years. What interest rate is needed for the future value of the CSB to be $15 000 at maturity?

b) Suppose that the interest rate from part a) was increased by 1%. What would be the future value of the CSB at maturity?

7. a) Predict which investment will earn the greater amount of interest over 5 years. Explain your prediction, and then verify it.

A. $1000 in a simple interest investment at 6%, paid semi-annually

B. $1000 in a simple interest investment at 6%, paid monthly

b) Is there an advantage if interest is paid more often? Explain.

c) Why might someone choose investment B over investment A?

8. a) Suzette plans to invest $10 000 and is researching the best deal online. The table below shows four investment options, with their interest rates and terms. Rank these options by their future values at maturity.

Investment Option	Simple Interest Rate	Term (years)
A	5% paid semi-annually	5
B	1.4% paid annually	10
C	1.9% paid quarterly	7
D	1.05% paid annually	8

b) Suppose that Suzette decided to withdraw her money 1.5 years before the end of the term. Would your ranking of the investment options change? Explain.

9. On July 1, Desiree deposited $3600 into a savings account that earns 2.5% simple interest, paid daily. On the same day, her sister Latoya deposited $3500 into a savings account that earns 3% simple interest, paid daily.

a) Who will have more money on December 31? How much more?

b) Determine the difference in the interest that the sisters will earn over the 6 months.

c) Compare their rates of return.

10. Shaun has been looking at houses. He has $10 000 that he wants to invest, hoping that he can end up with $15 000 to make a down payment on a house. He has an opportunity to invest at 6.5% simple interest, paid annually. How long will it take before Shaun can make a down payment of $15 000?

11. A bank is offering a simple interest rate of 3.2% for a guaranteed investment certificate with a 5-year term.

a) What principal would you need to invest if you wanted to have $20 000 at the end of the term?

b) How long would it take for the value of the GIC to be $25 000?

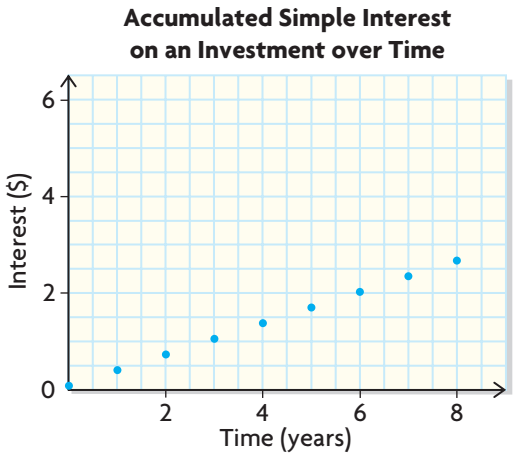
12. Lin invested $4700. After 8 years, the investment's value was $9400.

a) What was the annual simple interest rate?

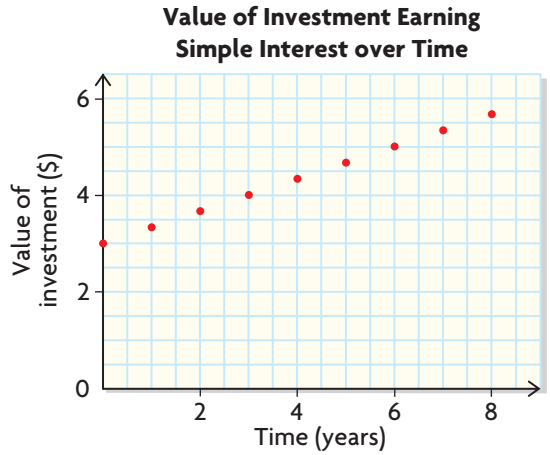
b) Suppose that the interest rate continued for another 8 years. What would be the value of the investment?

Closing

13. a) The simple interest earned on an investment represents a linear relation graphed against time. What would cause the slope of the graph to increase? Explain.



b) The value of this investment also represents a linear relation when graphed against time. Compare the graph in part a) with the graph below. How are these situations the same and how are they different?



Extending

14. At the beginning of each year, Graham purchases a $1000 Canada Savings Bond with a simple interest rate of 3.8%. After 5 years, what is the total value of Graham's CSBs?

15. Carole opened a savings account with $24 000. The account earned simple interest at 5.2%, paid daily. Carole closed her account after 86 days and withdrew all her money. How much money did she withdraw?

1.2 Exploring Compound Interest

YOU WILL NEED

- calculator
- spreadsheet software (optional)

GOAL

Compare simple interest with compound interest.

EXPLORE the Math

Guaranteed investment certificates (GICs) can earn either simple or compound interest. If a GIC earns simple interest annually, the same amount of interest is earned every year. If a GIC earns **compound interest** annually, the interest at the end of the first year is earned on the principal, but the interest at the end of the second year is earned on the principal plus the interest from the first year. Each year after that, the interest is earned on the principal plus all the accumulated interest from the previous years.

compound interest

The interest that is earned or paid on both the principal and the accumulated interest.

Both Ewan and Rena received a $1000 prize in a story-writing contest.

- Ewan bought a $1000 simple interest GIC with his prize money. It has a 5-year term and earns 3.6% paid annually.
- Rena bought a $1000 compound interest GIC with her prize money. It also has a 5-year term and earns 3.6% paid annually.

? How do the future values of Ewan's and Rena's investments compare at maturity?

Reflecting

A. With a partner, compare your answers and the strategies you used to determine the difference between the two investments at maturity.

B. Graph both investments on the same coordinate grid. How are the shapes of the graphs different? Explain why.

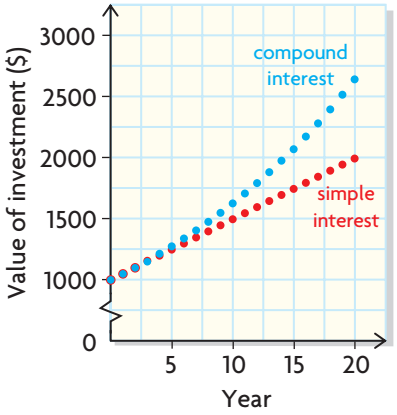
C. How much would Ewan need to invest at 3.6% simple interest to earn the same as Rena in 5 years?

In Summary

Key Ideas

- Compound interest is determined by applying the interest rate to the sum of the principal and any accumulated interest. Previously earned interest is reinvested over the course of the investment.
- If the same principal is invested in a compound interest account and a simple interest account, with the same interest rate for the same term, the compound interest investment will grow faster (non-linear) than the simple interest investment (linear). For example, the graphs show principal of $1000 invested over 20 years at 5% simple interest (red graph) and 5% compound interest (blue graph), both paid annually.

Comparing Simple and Compound Interest Investments at 5%



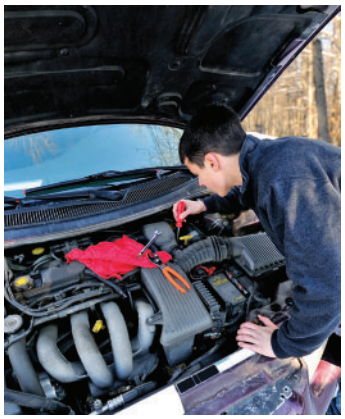
Need to Know

- Financial institutions pay compound interest on investments at regular equal intervals. If interest is paid annually, it is calculated at the end of the first year on the principal and then added to the principal. At the end of the second year, the interest is calculated on the balance at the end of the first year (principal plus interest earned from the previous year). This pattern continues every year until the end of the investment term.

FURTHER Your Understanding

1. Determine the difference in the interest earned at maturity on these two investments. Explain any discrepancies.
- Eve invested $3000 in a GIC for a term of 5 years with a simple interest rate of 4%, paid annually.
- Larry invested $3000 in a GIC for a term of 5 years with a compound interest rate of 4%, paid annually.

2. Sydney wants to open a savings account. He has $6500 to deposit. He intends to keep the account for 4 years and then use the money to rebuild the engine of his car. Which account should he choose? Justify your choice.
- **A.** 5.1% simple interest, paid weekly
- **B.** 4.8% compound interest, paid annually



3. The amount of interest that an investment earns is sometimes called the return on the investment.

a) Without determining the future value of each investment below, is it possible to predict, with confidence, which investment will have the greatest return? Explain.
- **A.** $6000 invested for 4 years at a compound interest rate of 1.2%
- **B.** $5000 invested for 5 years at a simple interest rate of 5%
- **C.** $4000 invested for 6 years at a compound interest rate of 6%

b) Determine which investment will have the greatest return.

1.3 Compound Interest: Future Value

YOU WILL NEED

- calculator
- spreadsheet software
- financial application on a graphing calculator

GOAL

Determine the future value of an investment that earns compound interest.

LEARN ABOUT the Math

EXPLORE...

- What is the next term in this pattern? How do you know?
  100, 150, 225, 337.5, 506.25, ...

compounded annually

When compound interest is determined or paid yearly.

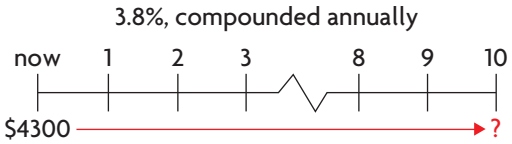
Yvonne earned $4300 in overtime on a carpentry job. She invested the money in a 10-year Canada Savings Bond that will earn 3.8% **compounded annually**. She decided to invest in a CSB, instead of keeping the money in a savings account, because the CSB will earn more interest.

? What is the future value of Yvonne's investment after 10 years?

EXAMPLE 1 Using reasoning to develop the compound interest formula

Deep's Solution

P is $4300.
r is 3.8% or 0.038, compounded annually.
t is 0, 1, 2, 3, ..., 10 years.



To organize my thinking, I recorded the information I knew in a timeline. Each space along the timeline represents one year. Each number represents the end of that year.

At the end of year 1:

$A = P(1 + rt)$

$A = 4300(1 + (0.038)(1))$
$A = 4300(1.038)$
$A = 4463.40$

I used the simple interest formula to determine the value of the investment at the end of year 1.

After year 2:

$A = 4463.40(1 + (0.038)(1))$

$A = 4463.40(1.038)$

$A = 4633.01$

I used the simple interest formula again to determine the value at the end of year 2. This time, however, the principal was the value at the end of year 1, $4463.40.

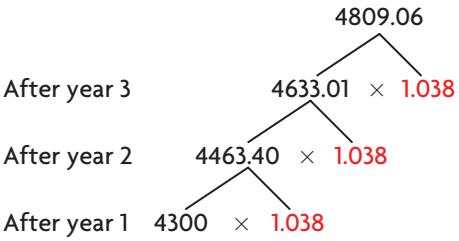
After year 3:

$A = 4633.01(1 + (0.038)(1))$

$A = 4633.01(1.038)$

$A = 4809.06$

I used the simple interest formula again to determine the value at the end of year 3. This time, the principal was the value at the end of year 2, $4633.01.



I used a diagram to show the pattern in the future value determinations for each year.

My diagram showed that I had multiplied the principal, $4300, by a factor of 1.038 three times to determine the value at the end of year 3.

I decided that I could extend the pattern to determine the future value at the end of year 10.

After 3 years:

$A = 4300(1.038)(1.038)(1.038)$

$A = 4300(1.038)^3$

$A = 4809.06$

After 10 years:

$A = 4300(1.038)^{10}$

$A = 6243.699\ldots$

The future value after 10 years is $6243.70.

I represented the pattern in an equation that shows the future value after 10 years of compounding. The future value is the product of the principal, $4300, and 1.038 raised to the power of the number of **compounding periods**, 10.

compounding period

The time over which interest is determined; interest can be compounded annually, semi-annually (every 6 months), quarterly (every 3 months), monthly, weekly, or daily.

$6243.699... = 4300(1.038)^{10}$
or
$6243.699... = 4300(1 + 0.038)^{10}$

I realized that 0.038 is the compound interest rate expressed as a decimal.

Let A represent the future value,
P represent the principal,
i represent the interest rate per compounding period,
and n represent the number of compounding periods:
$A = P(1 + i)^n$

I used the pattern to develop a general formula for determining the future value of any investment that earns compound interest.

Reflecting

A. Describe the pattern in the year-by-year calculations of the amount of Yvonne's investment.

B. The compound interest earned (I) on an investment at the end of any compounding period is the difference between the value of the investment at that time (A) and the original principal (P):

$$I = A - P$$

How can this relationship be represented symbolically using the variables I, A, P, i, and n?

C. For Yvonne's investment, the number of compounding periods in the term was the same as the number of years. Suppose that the interest had been compounded semi-annually. How many compounding periods would there have been at maturity? Explain.

APPLY the Math

EXAMPLE 2 Determining the future value of an investment with semi-annual compounding

Matt has invested a $23 000 inheritance in an account that earns 13.6%, compounded semi-annually. The interest rate is fixed for 10 years. Matt plans to use the money for a down payment on a house in 5 to 10 years.

a) What is the future value of the investment after 5 years? What is the future value after 10 years?

b) Compare the principal and the future values at 5 years and 10 years. What do you notice?

c) If the investment had earned simple interest, would the relationship between the principal and the future values have been the same? Explain.

Matt's Solution

a) Annual rate, $r = 13.6\%/\text{a}$

Interest rate over each compounding period,

$$i = \frac{13.6\%}{2}$$

$i = 6.8\%$/half year or 0.068

Since the interest rate is annual but the compounding period is semi-annual, I determined the semi-annual interest rate by dividing the annual rate by 2.

Term of 5 years:

Number of compounding periods, $n = (5)(2)$

$n = 10$

Multiplying the term in years by the number of times interest is earned each year gave me the number of compounding periods, n.

Principal, P, $= \$23\,000$

Future value after 5 years, $A_5 = P(1 + i)^n$

$A_5 = 23\,000(1 + 0.068)^{10}$

$A_5 = \$44\,405.87$

I used the compound interest formula to determine the future value of the investment after 5 years and after 10 years.

Term of 10 years:

Number of compounding periods, $n = (10)(2)$

$n = 20$

Future value after 10 years, $A_{10} = P(1 + i)^n$

$A_{10} = 23\,000(1 + 0.068)^{20}$

$A_{10} = \$85\,733.96$

b) Principal, $P = \$23\,000$

Future value after 5 years, $A_5 = \$44\,405.87$

Future value after 10 years, $A_{10} = \$85\,733.96$

After 5 years, the future value is just less than twice the principal. After 10 years, or double the time, the future value is more than triple the principal.

In the first 5 years, the investment earned \$21 405.87 in interest. In the next 5 years, it earned \$41 328.09 in interest. The difference in the interest earned in the two 5-year periods is due to the compounding of the interest over time.

c) No, the relationship would have been different.

Simple interest:

$I = Prt$

$I = 23\,000(0.136)(5)$

$I = 15\,640$

With simple interest, \$15 640 would have been earned after 5 years and $2 \cdot \$15\,640$ or \$31 280 would have been earned after 10 years. After 10 years of simple interest, the investment would have earned exactly twice as much interest as it would have earned after 5 years.

Your Turn

Suppose that Matt invested in an account earning 13.6%, compounded quarterly. Predict how the future values at 5 years and 10 years would change. Explain your prediction, and then verify it.

EXAMPLE 3 Determining the future value of investments with monthly compounding

Both Joli, age 50, and her daughter Lena, age 18, plan to invest $1500 in an account with an annual interest rate of 9%, compounded monthly.

a) If both women hold their investments until age 65, what will be the difference in the future values of their investments?

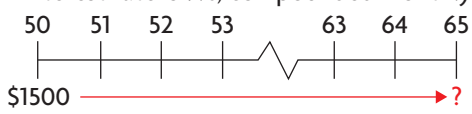
b) Lena's older step-brother Cody, age 34, also plans to invest $1500 at 9%, compounded monthly. Determine the future value of his investment at age 65.

Lena's Solution

a)

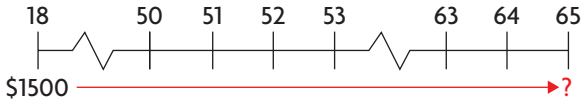
Joli's Investment (15 years)

interest rate is 9%, compounded monthly



My Investment (47 years)

interest rate is 9%, compounded monthly



I drew a timeline for each investment to organize the given information and visualize the problem.

I have about triple the amount of time for my investment to earn interest. I predicted that I will earn quite a bit more than three times the amount of interest, because the interest is compounded.

The annual interest rate is 9% or 0.09.

The monthly interest rate, i, is $\frac{0.09}{12}$ or 0.0075.

Number of compounding periods, n, for Joli's investment:

$(65 - 50)(12) = 180$

Number of compounding periods, n, for my investment:

$(65 - 18)(12) = 564$

Before I could use the future value formula for compound interest, I had to determine the number of compounding periods. To determine the number of compounding periods, I subtracted each age from 65 and then multiplied by 12, since compounding was 12 times a year.

Joli's Investment	My Investment
$A = P(1 + i)^n$ $A = 1500(1 + 0.0075)^{180}$ $A = 5757.064...$	$A = P(1 + i)^n$ $A = 1500(1 + 0.0075)^{564}$ $A = 101\,461.709...$

I used the compound interest formula to determine each future value.

I will earn almost $100 000 in interest ($101\,461.71 - 1500 \doteq 100\,000$), while my mother will earn only about $4300 in interest ($5757.06 - 1500 \doteq 4300$).

$101\,461.709... - 5757.064... = 95\,704.644...$

My future value is $95 704.64 greater.

b) $P = 1500$
$i = 0.0075$
$n = 31 \cdot 12$ or 372

$A = P(1 + i)^n$

$A = 1500(1 + 0.0075)^{372}$

$A = 24\ 168.61$

Cody's investment will have a future value of $24 168.61.

Although Cody's investment will have 31 years to grow (exactly halfway between 15 and 47 years), I predicted that his future value will be a lot less than halfway between the future values of the 15-year and 47-year investments (which is about $53 000) because of compound interest.

This amount seems reasonable, given my prediction.

Your Turn

When Lena invested her money, she knew that she was investing it for a long time. She also knew that banks offer investments at higher interest rates for longer terms, although there are usually more restrictions on when the money can be withdrawn. How much more would Lena earn if she invested $1500 for 47 years at an interest rate of 12%, compounded monthly?

EXAMPLE 4 Comparing interest on investments with different compounding periods

Céline wants to invest $3000 so that she can buy a new car in the next 5 years. Céline has the following investment options:

A. 4.8% compounded annually
B. 4.8% compounded semi-annually
C. 4.8% compounded monthly
D. 4.8% compounded weekly
E. 4.8% compounded daily

Compare the interest earned by each of these options for terms of 1 to 5 years.

Céline's Solution

The principal is $3000. The term is 1 year to 5 years.

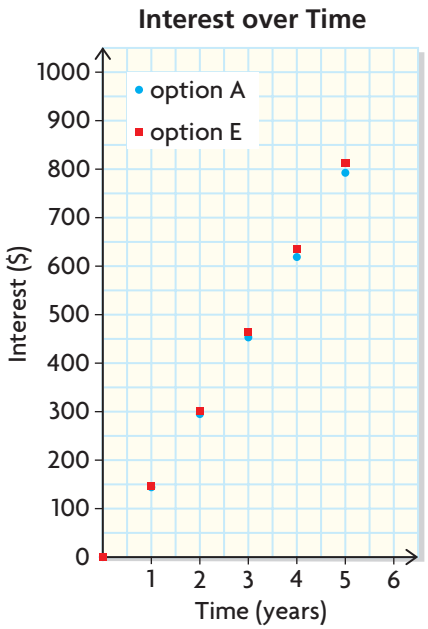
	Compounding Frequency	Compounding Periods per Year
A	annually	1
B	semi-annually	2
C	monthly	12
D	weekly	52
E	daily	365

I predicted that option E would have a greater future value for any length of term because it has the most frequent compounding. The more frequent the compounding, the greater the impact that the compounding will have.

I created a spreadsheet and used the formula $A = P(1 + i)^n$ to determine the future value of each option for 1 to 5 years.

I chose to have the decimal float, so the future values would be as exact as possible.

		A	B	C	D	E
1	**Principal ($)**	3000	3000	3000	3000	3000
2	**Interest Rate per Annum**	0.048	0.048	0.048	0.048	0.048
3	**Periods per Year**	1	2	12	52	365
4	**Value at End of Year**					
5	**0**	3000	3000	3000	3000	3000
6	**1**	3144	3145.728	3147.211	3147.442	3147.502
7	**2**	3294.912	3298.535	3301.645	3302.131	3302.256
8	**3**	3453.068	3458.765	3463.657	3464.422	3464.62
9	**4**	3618.815	3626.777	3633.620	3634.690	3634.966
10	**5**	3792.518	3802.952	3811.922	3813.325	3813.687



I graphed options A and E to compare them visually. I chose options A and E because yearly versus daily compounding would show the greatest difference over time.

I copied the rows of the spreadsheet that showed the future values and then subtracted $3000 (the principal) from each future value.

The graph of each option shows the interest earned increasing in a non-linear way over time. The points for each option grow farther apart over time. Daily compounding (option E) earns the most interest.

It does not matter how many years the investment is held. The more frequent the compounding, the greater the interest earned.

Your Turn

Which representation, the table of values or the graph, do you think better shows the effect of compounding frequency on an investment? Explain.

EXAMPLE 5 Estimating doubling times for investments

Both Berta and Kris invested $5000 by purchasing Canada Savings Bonds. Berta's CSB earns 8%, compounded annually, while Kris's CSB earns 9%, compounded annually.

a) Estimate the doubling time for each CSB.
b) Verify your estimates by determining the doubling time for each CSB.
c) Estimate the future value of an investment of $5000 that earns 8%, compounded annually, for 9, 18, and 27 years. How close are your estimates to the actual future values?

Percy's Solution

a) Berta's CSB:

$$\frac{72}{8} = 9$$

It will take about 9 years for Berta's CSB to double in value.

Kris's CSB:

$$\frac{72}{9} = 8$$

It will take about 8 years for Kris's CSBs to double in value.

Since both interest rates are compounded annually, I used the **Rule of 72** to estimate.

I divided 72 by the annual interest rate as a percent.

Rule of 72

A simple formula for estimating the doubling time of an investment; 72 is divided by the annual interest rate as a percent to estimate the doubling time of an investment in years.

The Rule of 72 is most accurate when the interest is compounded annually.

b) Berta's CSB:

The principal is $5000.
The annual interest rate is 8%.
The compounding frequency is annual, or 1 time per year.
The term (in years) is unknown.
The future value is double $5000, or $10 000.

I used the financial application on my calculator and entered these values to determine an exact doubling time for Berta's CSB. I solved for the term, in years.

The doubling time is 9.01 years, which is very close to my estimate of 9 years.

The value is 9.006..., which I rounded to two decimal places.

Kris's CSB:
The principal is $5000.
The annual interest rate is 9%.
The compounding frequency is annual, or 1 time per year.
The term (in years) is unknown.
The future value is double $5000, or $10 000.

The doubling time is 8.043... years, which is very close to the estimate of 8 years.

c) In about $\frac{72}{8}$ or 9 years, the investment doubles, so
- in 9 years, it will be about $10 000,
- in 18 years, it will be about $20 000, and
- in 27 years, it will be about $40 000.

After 9 years, the future value will be $9995.02.
After 18 years, the future value will be $19 980.10.
After 27 years, the future value will be $39 940.31.

I used the financial application on my calculator to determine the exact future values.

The Rule of 72 is less accurate when there is repeated doubling.

Your Turn

Use the Rule of 72 to estimate the doubling time for each investment. Then determine the doubling time. What do you notice about the effect of the compounding frequency on the accuracy of your estimate?

a) $5000 at 8%, compounded semi-annually
b) $5000 at 8%, compounded monthly
c) $5000 at 8%, compounded weekly
d) $5000 at 8%, compounded daily

In Summary

Key Ideas

- The future value of an investment that earns compound interest can be determined using the compound interest formula

$$A = P(1 + i)^n$$

where A is the future value, P is the principal, i is the interest rate per compounding period (expressed as a decimal), and n is the number of compounding periods.

- The more frequent the compounding and the longer the term, the greater the impact of the compounding on the principal and the greater the future value will be.

Need to Know

- When using the compound interest formula, use an exact value for i. For example, for an annual interest rate of 5% compounded monthly, substitute $\frac{0.05}{12}$ for i instead of the rounded value 0.004 16....
- Four common compounding frequencies are given in the table below. The table shows how the interest rate per compounding period (i) and the number of compounding periods (n) are determined.

Compounding Frequency	Times per Year	Interest Rate per Compounding Period (i)	Number of Compounding Periods (n)
annually	1	$i =$ annual interest rate	$n =$ number of years
semi-annually	2	$i = \frac{\text{annual interest rate}}{2}$	$n =$ (number of years)(2)
quarterly	4	$i = \frac{\text{annual interest rate}}{4}$	$n =$ (number of years)(4)
monthly	12	$i = \frac{\text{annual interest rate}}{12}$	$n =$ (number of years)(12)

- The total compound interest earned on an investment (I) after any compounding period can be determined using the formula

$$I = A - P \quad \text{or} \quad I = P[(1 + i)^n - 1]$$

- The Rule of 72 is a simple strategy for estimating doubling time. It is most accurate when the interest is compounded annually. For example, \$1000 invested at 3% interest, compounded annually, will double in value in about $\frac{72}{3}$ or 24 years; \$1000 invested at 6% will double in about $\frac{72}{6}$ or 12 years.

CHECK Your Understanding

1. Copy and complete the table.

Compound Interest Rate per Annum (%)	Compounding Frequency	Term	Interest Rate per Compounding Period, *i* (%)	Number of Compounding Periods, *n*
10.2	semi-annually	4 years		
4.1	monthly	6 years		
13.2	quarterly	7 years		
3.5	daily	9 months		

2. Determine the future value and the total interest earned for each investment.

a) $520 invested for 8 years at 4.5% compounded monthly

b) $1400 invested for 15 years at 8.6% compounded semi-annually

PRACTISING

3. For each investment,

i) use the Rule of 72 to estimate the doubling time and then determine the doubling time.

ii) determine the future value and the total interest earned.

	Principal (*P*) ($)	Rate of Compound Interest per Annum (%)	Compounding Frequency	Term (years)
a)	7 000	6.8	annually	35
b)	850	9.2	monthly	20
c)	12 500	15.6	weekly	5
d)	40 000	2.7	semi-annually	8

4. When Willa was born, her grandparents set up two investments of $3000 for her. One earns 9%, compounded annually; the other earns 9%, compounded monthly.

a) Willa is now 18. Determine the current value of each investment.

b) Graph the interest earned over time for both investments on the same grid. Plot at least five points for each investment.

c) How does the compounding frequency affect the growth of interest?

5. Parker wanted to buy a new motorcycle but he had only $6000, half the amount he needed.

a) Estimate when Parker could buy the motorcycle if he invested his money at 4.8%, compounded annually. Verify your estimate.

b) Estimate how much sooner he could buy the motorcycle if his investment earned 7.2%, compounded annually. Verify your estimate.

6. Trust funds are investments that are set up for a specific purpose. A local business invested $250 000 in a charitable trust fund so that a school can offer scholarships. The interest rate is 3.8%, compounded semi-annually. Only the interest earned can be used to provide the scholarships. How much is available from the trust fund for scholarships each year?

7. Suppose that you are searching online for the best interest rates on a GIC. You find these rates:

- Bank A offers 6.6%, compounded annually.
- Bank B offers 6.55%, compounded semi-annually.
- Bank C offers 6.5%, compounded quarterly.

Rank these rates from greatest to least return on an investment of $20 000 for a term of 2 years.

8. Estimate how long it would take for $1000 to grow to $16 000 at each interest rate, compounded annually.

a) 6% **b)** 12%

9. Angie deposited some money into an account with a fixed rate of interest, compounded annually, for 3 years. The growth of the investment is shown in the table below. What is the annual rate of interest? What was the principal that Angie invested?

End of Year	Value of Investment ($)
1	852.00
2	907.38
3	966.36

10. Solomon bought a $40 000 corporate bond (an investment in the form of a loan to a company that earns interest). The bond earns 4.8%, compounded semi-annually. After 4 years, the interest rate changed to 6%, compounded annually. Determine the value of Solomon's investment after 6 years.

11. On Freda's 16th birthday, she invested $1500 in an account that earns 9%, compounded semi-annually. On her 20th birthday, she moved her investment to an account that paid 11%, compounded monthly. Determine the value of her account on her 22nd birthday.

Term (years)	Rate (%)
1	1.35
2	1.65
3	1.90
4	2.15
5	2.65
6	2.70
7	2.85
8	2.90
9	3.00
10	3.25

12. Lenny has $5000 to invest and is looking at different GICs, as shown in the table to the left. These GICs cannot be redeemed until their maturity.
- **a)** Why do you think the interest rates increase as the term increases?
- **b)** Lenny cannot decide whether to invest $5000 for 10 years or to invest $5000 for 5 years and then reinvest for another 5 years.
  - **i)** Compare the future values of each option. What assumptions are you making?
  - **ii)** What are the advantages and disadvantages of each option?

Closing

13. Compare simple and compound interest investments by describing what they have in common and what is unique to each.

Extending

14. For each of the past 5 years, Purleen has purchased a $500 Canada Savings Bond on the same date. The interest rate on all the CSBs is 2.9%, compounded semi-annually.
- **a)** What is the current total value of Purleen's CSBs?
- **b)** If Purleen continues this pattern of purchase, what will be the value of her CSBs after another 5 years?

15. This year and on subsequent alternating years, Terry plans to invest $900 in a savings account that earns 11.2%, compounded quarterly. What is the value of his savings immediately after he has made his fourth investment?

History | Connection

Interest Rates by the Decade

Interest rates depend on many economic factors and vary over the years. In Roman times, interest rates typically ranged from 4% to 12% and were paid monthly. It was not rare, however, to have an interest rate that was a multiple of 12, such as 24% or 48%!

Today, the interest rates that are offered by financial institutions, such as banks, are influenced by the economy, both in Canada and the world. On the other hand, interest rates can be used to influence the economy. Interest rates can be lowered to encourage people to borrow and spend in order to stimulate the economy, or they can be raised to encourage saving.

Over the past century, interest rates in Canada have varied from 0% to almost 20%. Canadian banks set their interest rates according to the rate set by the Bank of Canada.

A. Research the Bank of Canada prime interest rates for one decade.

B. Choose the lowest and highest interest rates within the decade. Determine how much more a compound interest investment of $10 000 would earn at the higher rate, compared with the lower rate.

Applying Problem-Solving Strategies

Saving for Retirement

Imagine that you have just started your first full-time job. You have set the financial goal of saving $1 000 000 for your retirement.

The Game

A. Decide what year you will start acting on the financial goal.

B. Research to determine the current interest rates on savings accounts, GICs, and Canada Savings Bonds. Assume these are the rates that will be available to you when you start acting on your goal.

C. Choose an investment and make an initial deposit of $5000.

D. Determine the future value of your initial investment when the investment matures, or after one year, if the investment has no maturity date.

E. Roll a standard die. If the result is 1 or 2, the current interest rates on all investments decrease by 1%. If the result is 3 or 4, interest rates stay the same. If the result is 5 or 6, interest rates increase by 1%.

F. Based on the new rates, choose an investment and invest whatever you have from the previous investment, plus another $5000 for each year that has passed since your last investment.

G. Repeat steps C through F until you have $1 000 000. How old will you be when you reach your goal?

The Strategy

H. Describe the investment strategy you used to determine the youngest age you will be when you reach your goal.

I. Which factor seems to be more important: the strategy you are using, or the outcome of the die roll? Explain.

Creating a Variation of the Game

J. Can you make the game more realistic? Describe any modifications you could make, and explain the effect you expect from your modifications.

K. Play the game again, using your modified rules. Does the modified game seem to increase or decrease the age at which you reach the goal?

YOU WILL NEED

- calculator
- spreadsheet software or financial application on graphing calculator
- standard die

1.4 Compound Interest: Present Value

YOU WILL NEED

- calculator
- spreadsheet software
- financial application on a graphing calculator

EXPLORE...

- Nora wants to have $5000 at the end of 3 years for college. All the interest rates that she has found are less than 10%. Determine two possible amounts that she might invest to reach her goal of $5000.

GOAL

Determine the principal or present value of an investment, given its future value and compound interest rate.

INVESTIGATE the Math

In 5 years, after graduating from college, Cal wants to spend a year travelling in Canada's three territories. He plans to start in Yukon and then travel east to the Northwest Territories and Nunavut. Cal has determined that he will need at least $15 000 for his trip. To reach this goal, he wants to invest money now. He has chosen a GIC at 7%, compounded annually.

? How much does Cal need to invest now so that he will have $15 000 in 5 years?

A. The future value of an investment that earns compound interest, given the principal or **present value**, interest rate, and compounding frequency, can be determined using the compound interest formula:

$$A = P(1 + i)^n$$

How could you use this formula to determine the present value, given the future value, interest rate, and compounding frequency?

present value

The amount that must be invested now to result in a specific future value in a certain time at a given interest rate.

B. Determine how much principal Cal needs to invest now, at 7% compounded annually, to have a future value of $15 000 in 5 years.

Reflecting

C. How could you verify your answer to part B?

D. **a)** Why might someone want to know the ratio of the future value of an investment to its present value?

b) How could you use the compound interest formula to determine this ratio?

APPLY the Math

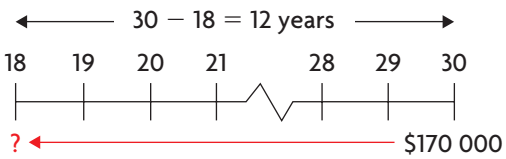
EXAMPLE 1 Determining the present value of investments earning compound interest

Ginny is 18 years old. She has inherited some money from a relative. Ginny wants to invest some of the money so that she can buy a home in Milk River, Alberta, when she turns 30. She estimates that she will need about $170 000 to buy a home.

a) How much does she have to invest now, at 6.5% compounded annually?
b) What is the ratio of future value to present value for Ginny's investment?
c) How would the ratio change if the interest rate decreased to 6% but was compounded semi-annually?

Ginny's Solution

a) interest rate is 6.5%, compounded annually



I made a timeline to record and organize the given information and visualize the problem.

I could see that I needed to know the principal, or present value, that would grow to $170 000 as it earned 6.5% interest. The term of the investment is from now until I am 30. Since I am 18, this is 12 years.

$A = 170\,000$

$i = \frac{0.065}{1}$

$n = 12$

$A = P(1 + i)^n$

$P = \frac{A}{(1 + i)^n}$

$P = \frac{170\,000}{(1.065)^{12}}$

$P = 79\,846.085...$

I need to invest \$79 846.09.

Since the compounding period is annual, the annual interest rate is also the interest rate per compounding period, and the number of years in the term is also the number of compounding periods.

I rewrote the formula for future value by isolating the present value (P), because I knew the future value (A), the interest rate per annual compounding period (i), and the number of compounding periods over the term.

b) $\frac{A}{P} = \frac{170\,000}{79\,846.09}$

$\frac{A}{P} = 2.129...$

The ratio of future value to present value for 12 years is approximately 2.13.

I determined the ratio of future value to present value, using the values from part a). I expressed the ratio as a decimal, to two decimal places.

c) $i = \frac{0.06}{2}$ or 0.03

$n = 2(12)$ or 24

$\frac{A}{P} = (1 + i)^n$

$\frac{A}{P} = (1.03)^{24}$

$\frac{A}{P} = 2.032...$

The ratio of future value to present value for 12 years is approximately 2.03. The 0.5% drop in the interest rate had more effect than doubling the compounding frequency, so the overall ratio decreased.

Instead of determining the present value again and then determining the ratio, I rewrote the compound interest formula by isolating the $\frac{A}{P}$ ratio.

Using the $\frac{A}{P}$ ratio to compare these investments makes sense since the term is the same.

Your Turn

Ginny figures that if she waited another 12 years, she would need to invest only half the present value at 6.5% compounded annually. Do you agree? Explain.

EXAMPLE 2 Determining the present value of an investment that is compounded quarterly

Agnes and Bill are musicians. They have researched the costs to set up a small recording studio. They estimate that $40 000 will pay for the soundproofing, recording equipment, and computer hardware and software that they need. They plan to set up the studio in 3 years and have invested money at 9.6%, compounded quarterly, to save for it.

a) How much money should they have invested?
b) How much interest will they earn over the term of their investment?

Gerald's Solution

a) interest rate is 9.6%, compounded quarterly

My timeline helped me visualize the problem and decide what to do.

I could see that I needed to know the present value that would grow to $40 000 as it earned 9.6% interest, compounded quarterly, for 3 years.

$A = 40\,000$

$i = \frac{0.096}{4}$ or 0.024

$n = 3(4)$ or 12

$P = \frac{A}{(1 + i)^n}$

$P = \frac{40000}{(1 + 0.024)^{12}}$

$P = 30\,092.655...$

Since the interest is compounded quarterly,
- I divided the per annum rate by 4 to determine the quarterly interest rate, and
- I multiplied the number of years in the term by 4 to determine the number of compounding periods.

I used the present value version of the compound interest formula because I knew the future value, interest rate, and number of compounding periods.

They should have invested $30 092.66.

b) $I = A - P$

$I = 40\,000 - 30\,092.66$

$I = 9907.34$

For total interest earned, I subtracted the present value from the future value.

They will earn $9907.34 in interest.

Your Turn

Would twice the present value be needed if the compounding frequency was half as often? Explain.

EXAMPLE 3 Determining an unknown interest rate and unknown term

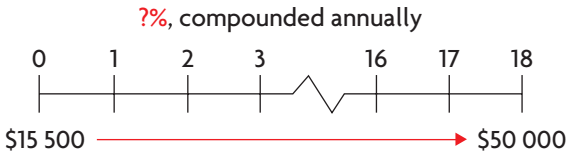
Laura has invested $15 500 in a Registered Education Savings Plan (RESP). She wants her investment to grow to at least $50 000 by the time her newborn enters university, in 18 years.

a) What interest rate, compounded annually, will result in a future value of $50 000? Round your answer to two decimal places.

b) Suppose that Laura wants her $15 500 to grow to at least $60 000 at the interest rate from part a). How long will this take?

Frank's Solution

a)



I sketched a timeline to record and organize the known information and to visualize the problem. From my timeline, I could see that I needed to know the minimum annual interest rate, compounded annually, that is required for $15 500 to grow to $50 000 over 18 years.

$A = P(1 + i)^n$

$A = 50\,000$

$P = 15\,500$

$n = 18(1)$

$50\,000 = 15\,500(1 + i)^{18}$

$\frac{50\,000}{15\,500} = (1 + i)^{18}$

I substituted the known values into the compound interest formula and solved for i.

$\sqrt[18]{\frac{50\,000}{15\,500}} = 1 + i$

$1.0672... = 1 + i$

$0.0672... = i$

The interest rate needed to make $15 500 grow to $50 000 is 6.72%.

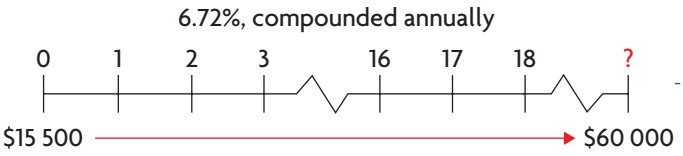
The present value is $15 500.
The annual interest rate is unknown.
The compounding frequency is annual or 1 time per year.
The term (in years) is 18.
The future value is $50 000.

I verified my answer using the financial application on my calculator.

I entered these values to determine the annual interest rate.

Interest rate = 0.067 22... or 0.0672
An interest rate of at least 6.72%, compounded annually, would give the desired future value.

b)



The present value is \$15 500.
The interest rate is 6.72%.
The compounding frequency is annual or 1 time per year.
The term (in years) is unknown.
The future value is \$60 000.

Term = 20.810...

It will take 21 years for \$15 500 to grow to at least \$60 000 at the interest rate from part a), since compounding is annual.

I revised my timeline.

I knew that more than 18 years would be needed for \$15 500 to grow to \$60 000, since \$15 500 grew to \$50 000 in 18 years at the same interest rate.

I had planned to use the compound interest formula, but realized that I couldn't solve this equation for n: $\frac{A}{P} = (1 + i)^n$

I used my financial application to avoid solving for an unknown that is an exponent. I entered these values to determine the term.

No interest will be paid on the last 0.810 year.

Your Turn

Could Frank have solved either of these problems if he had not known the compounding frequency that was expected? Explain.

In Summary

Key Idea

- The present value of an investment that earns compound interest can be determined using the formula

$$P = \frac{A}{(1 + i)^n}$$

where P is the present value (or principal), A is the amount (or future value), i is the interest rate per compounding period (expressed as a decimal), and n is the number of compounding periods.

Need to Know

- Any equivalent form of the compound interest formula may be used to solve a compound interest problem.

$A = P(1 + i)^n$ $\quad P = \frac{A}{(1 + i)^n}$ $\quad \frac{A}{P} = (1 + i)^n$

- To compare investments, usually with the same term or principal, the ratio of the future value to the present value can be determined using the form: $\frac{A}{P} = (1 + i)^n$
- Using a formula, using the financial application on a graphing calculator, and using spreadsheet software are all valid strategies for solving a compound interest problem.

CHECK Your Understanding

1. Predict which investment will require a greater present value to be invested. Explain your prediction, and then verify it.

A. Future value of $10 000 at 5%, compounded monthly, for 10 years

B. Future value of $10 000 at 5%, compounded quarterly, for 10 years

2. **a)** Determine the future value to present value ratio for both investments in question 1.

b) Would an investment with a future value of $10 000 at 6%, compounded annually, for 10 years have a higher or lower ratio? Explain.

3. Complete the table.

Future Value (amount in $)	Present Value (principal in $)	Interest Rate per Annum (%)	Compounding Frequency	Investment Term (years)
2 500.00	?	7.8	annually	8
3 500.00	2000.00	?	semi-annually	5
11 000.00	?	2.4	quarterly	12
100 000.00	609.35	13.6	annually	?
23 500.00	16 150.00	?	monthly	2

4. Mac plans to retire in 20 years, when he is 55. He estimates that he will need $250 000 to live on, until he is eligible for his pension.

a) How much money should he invest now, at 8.5% compounded annually, to meet his goal?

b) How much interest will he earn in 20 years?

PRACTISING

5. Joseppie is planning to buy a new snowmobile in 2 years. He intends to spend no more than $17 000. He has $9000 to invest in an account that compounds interest quarterly.

a) What rate of interest will Joseppie need to find in order to meet his goal? Is his plan reasonable?

b) Suppose that he manages to find an interest rate of 12%, compounded quarterly. How long will it take him to save enough money?

6. Claire wants a down payment of $17 500 to buy a house in 10 years, when she turns 30. Her bank offers her an investment with 5.6% interest, compounded semi-annually. What present value will she need to invest now?

7. Sasha predicts that she will need $24 000 to remodel her carpentry workshop in 6 years. She has found three investment options to consider:

A. 4.80%, compounded annually
B. 4.75%, compounded semi-annually
C. 4.70%, compounded quarterly

a) Compare the rates of return for these three options. Which option should she choose? Why?
b) How much interest will she earn?

8. Choose one of these investments:

A. $15 000 GIC that earns 3.8%, compounded annually, for 10 years
B. $26 000 CSB that earns 6.2% for the first 5 years, compounded semi-annually
C. $8000 investment savings account that earns 4.1%, compounded quarterly, for 8 years

a) Graph the value of the investment ($) against time (years).
b) Change either the interest rate or the principal. Graph the value of the new investment against time on the same grid.
c) Make another change to the same variable you changed in part b). Graph the value of the new investment against time on the same grid.
d) How did the changes in the variable affect the shape of the graph?

9. In 40 years, Blake wants to have $1 000 000. He plans to invest less than $10 000 now. Which of these investment options would allow him to invest the least and still meet his goal? Justify your choice.

A. 12.6%, compounded annually
B. 11.9%, compounded semi-annually
C. 13.2%, compounded quarterly
D. 11.53%, compounded weekly

10. Franco invested money at 6.9%, compounded annually, while David invested money at 6.9%, compounded monthly. After 30 years, each investment is worth $25 000. Who made the greater original investment, and by how much was it greater?

11. Lucy is investing $3000. She wants it to grow to $7000 in 10 years.

a) What annual rate of interest, compounded quarterly, does Lucy need to meet her goal? Round your answer to two decimal places.
b) What is the ratio of future value to present value for Lucy's investment? Predict whether this ratio would increase or decrease if Lucy invested $3000 at the same interest rate, but compounded annually. Explain your prediction, and verify it.

12. Daniel has a savings account that earns interest at 5.3%, compounded monthly. He has not made any deposits or withdrawals in the past 9 months. If there is $4765.30 in the account today, how much interest has the account earned in the past 9 months?

13. Ben would like to send his parents on a $15 000 safari for their 35th wedding anniversary in 10 years. He has the opportunity to invest in a GIC that earns 5.5%, compounded semi-annually. His brother and sister have agreed to split the cost of the GIC with him. How much will each sibling contribute to the cost of the GIC?

Closing

14. Imagine that you are a financial advisor. You have a client who knows nothing about investments. Explain the key features that your client should look for in a fixed-interest investment opportunity. Use the terms *present value*, *future value*, *simple interest*, *compound interest*, *interest rate*, *compounding frequency*, and *term* in your explanation.

Extending

15. What annual interest rate, compounded quarterly, would enable an investment to triple every 12 years? Round your answer to two decimal places.

16. **a)** What is the future value of $1000 if invested for 1 year at 5%, compounded annually?

b) What interest rate would result in the future value from part a) for each compounding frequency?

i) semi-annually **ii)** quarterly **iii)** monthly

c) What advantage might there be in choosing a lower interest rate that is compounded more frequently?

1 Mid-Chapter Review

FREQUENTLY ASKED Questions

Q: Why would someone choose an investment that paid compound interest over an investment that paid simple interest, assuming that the principal, interest rate, and term are the same?

A: Simple interest is earned only on the principal of the investment, while compound interest is earned on the principal and any accumulated interest. So, when the principal, interest rate, and term are the same, a compound interest investment will earn more interest than a simple interest investment. For example, consider both investments below.

Sample Simple Interest Investment
P = \$2000, r = 8%, t = 3 years

Term (year)	Value at Start of Year ($)	Interest Rate	Interest Earned ($)	Value at End of Year ($)
1	2000	0.08	160	2160
2	2160	0.08	160	2320
3	2320	0.08	160	**2480**

Sample Annual Compound Interest Investment
P = \$2000, i = 8%, n = 3

Term (year)	Value at Start of Year ($)	Interest Rate	Interest Earned ($)	Value at End of Year ($)
1	2000.00	0.08	160.00	2160.00
2	2160.00	0.08	172.80	2332.80
3	2332.80	0.08	186.62	**2519.42**

The compound interest investment earns \$39.42 more interest than the simple interest investment after 3 years.

Study *Aid*
- See Lessons 1.1 and 1.2.
- Try Mid-Chapter Review Questions 1 to 3.

Study Aid
- See Lessons 1.3 and 1.4.
- Try Mid-Chapter Review Questions 4 to 9.

Q: How do you determine what values to substitute for the variables *n* and *i* when using the compound interest formula,

$$A = P(1 + i)^n$$

A: The number of compounding periods (n) is determined by using the compounding frequency and the term (t). The interest rate per compounding period (i) is the quotient of the annual interest rate (r) and the compounding frequency. Use an exact value for i so there is no rounding error. For example, when \$1000 is invested at 5% for 10 years:

Semi-annual compounding:

$i = \frac{r}{2}$ $\qquad n = t(2)$

$i = \frac{0.05}{2}$ $\qquad n = 10(2)$

$i = \mathbf{0.025}$ $\qquad n = \mathbf{20}$

$A = P(1 + i)^n$

$A = 1000(1 + \mathbf{0.025})^{\mathbf{20}}$

Monthly compounding:

$i = \frac{r}{12}$ $\qquad n = t(12)$

$i = \frac{\mathbf{0.05}}{\mathbf{12}}$ $\qquad n = 10(12)$

$n = \mathbf{120}$

$A = P(1 + i)^n$

$A = 1000\left(1 + \frac{\mathbf{0.05}}{\mathbf{12}}\right)^{\mathbf{120}}$

Study Aid
- See Lessons 1.3 and 1.4.
- Try Mid-Chapter Review Questions 4 to 10.

Q: How do you know what form of the compound interest formula to use when solving a problem?

A: You can use any equivalent form of the compound interest formula, but you might prefer to use a form that simplifies your calculations. For example, when \$1000 is invested at 5%, compounded semi-annually, for 10 years, it grows to \$1638.62:

Determining the present value, when future value, annual interest rate, compounding frequency, and term are known:

$P = \frac{A}{(1 + i)^n}$

$P = \frac{1638.62}{(1 + \frac{0.05}{2})^{2(10)}}$

$P = \frac{1638.62}{(1 + \frac{0.05}{2})^{20}}$

$P = \$1000$

Determining the annual interest rate, when future value, present value, compounding frequency, and term are known:

$\frac{A}{P} = (1 + i)^n$

$\frac{1638.62}{1000} = (1 + i)^{20}$

$1.638\,62^{\frac{1}{20}} = [(1 + i)^{20}]^{\frac{1}{20}}$

$1.025\,000\,111... = 1 + i$

$0.025\,000\,111... = i$

$r = 2(0.025\,000\,111...)$

$r = 0.050...$ or 5%

You can also use a financial application to determine any unknown variable in a compound interest problem situation if you know the other variables. This is recommended when determining the term of the investment.

PRACTISING

Lesson 1.1

1. Paula earned $27.54 in simple interest by investing $450. The interest rate was 2.04%. For how many years did she hold the investment?

2. **a)** For how long would $6000 need to be invested, at 6.4% simple interest, to earn $1200 in interest?

b) How long would it take if the interest for part a) was paid yearly?

c) How long would it take if the interest was paid quarterly?

Lesson 1.2

3. Brad and Katherine deposited $5000 in two separate accounts when their baby was born. Katherine's account earns compound interest at 4.87%, paid annually. Brad's account earns simple interest at 5.5%. The investments will mature when their child turns 20.

a) What will each account be worth at maturity?

b) Graph both investments on the same coordinate grid.

c) What does the intersection point of the two graphs represent?

Lesson 1.3

4. Ron purchased a 10-year GIC for $3000. The GIC earns 5.6% interest, compounded annually.

a) What will be the future value of the GIC at maturity?

b) Estimate how long it will take for the GIC to be worth at least $12 000.

c) Predict what would happen to the future value of the GIC in each situation below. Explain your prediction, and then verify it.

i) The compounding frequency is monthly.

ii) The interest rate is 2.8%, compounded semi-annually.

d) What minimum interest rate, with daily compounding, would be needed to have a future value that is $100 greater than the future value you determined in part a)?

5. An alumnus of a local high school donated $50 000 to the school. The amount was invested for 3 years at 7.75%, compounded quarterly. The school has agreed to use only the interest earned on the investment to buy sports equipment. How much money will be available for sports equipment at the end of the investment's term?

6. Rank the following changes by their impact on the future value of $1000 invested at 5%, compounded annually, for 5 years:

A. Increasing the principal to $1050

B. Increasing the interest rate to 6%

C. Increasing the compounding frequency to monthly

D. Increasing the term to 6 years

Lesson 1.4

7. **a)** How much should Desiree invest at 6%, compounded monthly, to have $10 000 in 3 years?

b) How much should Desiree invest if the compounding period is semi-annual?

8. On Petra's 22nd birthday, she received a gift of $11 000. This was the future value of an investment that was made when she was born.

a) How much was invested 22 years ago

i) if the interest rate was 7.2%, compounded annually?

ii) if the interest rate was 7.2%, compounded semi-annually?

b) Suppose that $11 000 had been invested at 7.2% interest, compounded monthly, for 22 years. Would the ratio of future value to present value be higher or lower than the ratio for the original investment? Justify your answer.

9. An investment of $400 grew to $625 in 10 years. What was the annual interest rate if the interest was compounded monthly?

10. An investment of $250 grew to $1000 at 6% interest, compounded semi-annually. Estimate how long it took for the investment to grow, and then verify your estimate.

1.5 Investments Involving Regular Payments

YOU WILL NEED

- financial application on a graphing calculator or spreadsheet
- spreadsheet software

EXPLORE...

- Indu has been depositing $200 into a savings account at the end of every month. The interest rate on her minimum monthly balance is 6%, compounded monthly. She now has between $1000 and $1500. How long has she been depositing money into the account?

GOAL

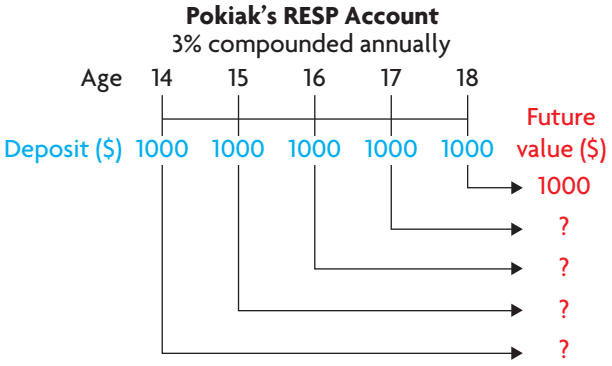
Determine the future value of an investment that earns compound interest involving regular payments.

INVESTIGATE the Math

Pokiak is now 18 years old, and he needs money for his post-secondary education. On his 14th birthday, his family deposited $1000 into a Registered Education Savings Plan (RESP) at 3% interest, compounded annually. Since then, Pokiak has deposited $1000 of his own money, earned by working part-time, into the account each year.

? How much money is in Pokiak's RESP account, and how much interest has it earned altogether?

A. Each of the five $1000 deposits could be thought of as a separate investment, as shown on the timeline below. Each deposit earns interest at the same interest rate. Will each deposit earn the same amount of interest? Explain.



B. For how many compounding periods does each deposit earn interest?

C. Determine the future value of each deposit.

D. Determine the current value of Pokiak's RESP account. What strategy did you use?

E. How much interest did Pokiak's RESP account earn altogether?

Reflecting

F. How is determining the future value of an investment involving regular deposits like determining the future value of a single deposit? How is it different?

G. How is determining the total interest earned on an investment involving regular deposits different from determining the total interest earned on a single deposit?

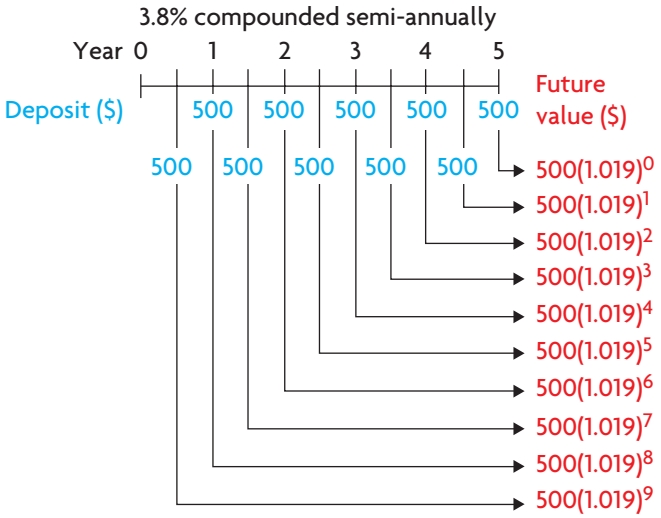
APPLY the Math

EXAMPLE 1 Determining the future value of an investment involving regular deposits

Darva is saving for a trip to Australia in 5 years. She plans to work on a student visa while she is there, so she needs only enough money for a return flight and her expenses until she finds a job. She deposits $500 into her savings account at the end of each 6-month period from what she earns as a server. The account earns 3.8%, compounded semi-annually. How much money will be in the account at the end of 5 years? How much of this money will be earned interest?

Darva's Solution: Using a spreadsheet

I drew a timeline to show the future value of each of the $500 deposits that I made at the end of each 6-month period for 5 years.



I could see that I needed to do 10 calculations and then determine the sum.

$A = P(1 + i)^n$

I represented each deposit's future value, A, using $P = \$500$, $i = \frac{0.038}{2}$, or 0.019, and the number of compounding periods for each deposit, n.

	A	B	C	D
1	**Year**	**Number of Compounding Periods (*n*)**	**Deposit ($)**	**Future Value ($)**
2	0.5	9	500	592.2944
3	1	8	500	581.2507
4	1.5	7	500	570.4128
5	2	6	500	559.7771
6	2.5	5	500	549.3396
7	3	4	500	539.0968
8	3.5	3	500	529.0449
9	4	2	500	519.1805
10	4.5	1	500	509.5
11	5	0	500	500
12		Totals	5000	5449.897

It made sense to use a spreadsheet because there are a lot of calculations.

In column D, I used the expression $500(1.019)^n$ to create the spreadsheet formula for the future value of each deposit.

In column B, I entered the value of n for each deposit.

In cell D12, I entered a formula to determine the future value of the investment (the sum of the future values of the 10 deposits).

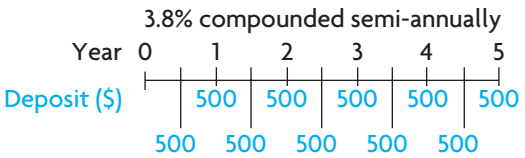
In cell C12, I entered a formula to determine the sum of the 10 deposits.

Interest earned $= 5449.90 - 5000$
Interest earned $= 449.90$

To determine the interest earned, I subtracted the total of the deposits from the future value of the investment.

There will be $5449.90 in the account at the end of 5 years, and $449.90 of this will be interest.

Len's Solution: Using a financial application



I drew a timeline to represent the investment. I could see that I needed to determine the future value of the ten $500 deposits, each earning interest at the same rate but for a different number of compounding periods.

The regular payment amount is $500.
The payment frequency is semi-annual, or 2 times per year.
The number of payments is 10.
The payments are made at the end of each payment period.
The annual interest rate is 3.8%.
The compounding frequency is semi-annual, or 2 times per year.
The future value is unknown.

I decided to use the financial application on my calculator.

Future value $= 5449.896...$

Interest earned $= 5449.896... - 10(500)$
Interest earned $= 449.896...$

I determined the total interest earned by subtracting the total payments from the future value.

There will be $5449.90 in the account at the end of 5 years, and $449.90 of this will be interest.

$5449.90 seems reasonable because deposits of $500 made twice a year for 5 years would total $5000, without counting the interest.

Your Turn

Suppose that Darva's deposits were only $400 every 6 months instead of $500 every 6 months, and that the interest rate on her account remains 3.8%, compounded semi-annually. At the end of 5 years, how much less would the future value of the account be? How much interest would Darva earn?

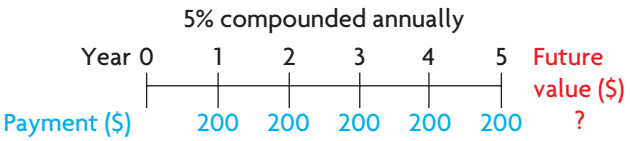
EXAMPLE 2 Comparing a regular payment investment with a single payment investment

Adam made a $200 payment at the end of each year into an investment that earned 5%, compounded annually. Blake made a single investment at 5%, compounded annually. At the end of 5 years, their future values were equal.

a) What was their future value?
b) What principal amount did Blake invest 5 years ago?
c) Who earned more interest? Why?

Eva's Solution

a) Adam's investment:



I drew a timeline to help me visualize the problem.

I could see that there would be a lot of calculations if I determined the future value of each payment and then determined the sum, so I decided to use technology.

The regular payment amount is $200.
The payment frequency is annual, or 1 time per year.
The number of payments is 5.
The payments are made at the end of each payment period.
The annual interest rate is 5%.
The compounding frequency is annual, or 1 time per year.
The future value is unknown.

I used the financial application in my spreadsheet software to determine the future value of the investment.

The future value of both investments was $1105.126....

b) Blake's investment:

The present value is unknown.
The annual interest rate is 5%.
The compounding frequency is annual, or 1 time per year.
The term (in years) is 5.
The future value is \$1105.126....

Present value = 865.895...
Five years ago, Blake invested \$865.90.

I predicted that Blake's investment would be less than the total amount Adam invested (\$1000), because Blake's entire investment had 5 years to earn interest.

I used the financial application again, but this time I used it to determine the present value of Blake's single payment investment.

c) Adam's investment:

Interest earned = Future value − Present value
Interest earned = 1105.126... − 5(200)
Interest earned = 105.126...

Blake's investment:

Interest earned = Future value − Present value
Interest earned = 1105.126... − 865.895...
Interest earned = 239.230...

Difference in interest earned = 239.230... − 105.126...

Blake earned \$134.10 more interest.

Blake invested less but earned more interest, even though the interest rate, compounding frequency, and term were the same.

Blake's principal of \$865.90 earned 5% interest, compounded annually, for 5 years. In contrast, only \$200 of Adam's investment earned 5% interest, compounded annually, for 5 years. The second \$200 payment earned interest for only 4 years, the third \$200 payment earned interest for only 3 years, and so on.

Adam's investment was like a series of individual investments, each with a term that was 1 year shorter than the term before it.

Your Turn

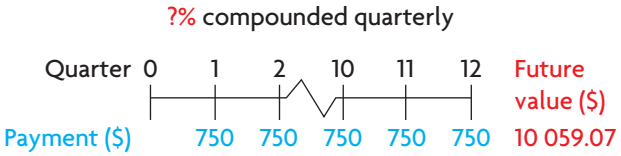
a) Suppose that Blake had invested the same total amount as Adam, but as a single investment. Predict how their future values would compare. Explain and then verify your prediction.

b) What rate of interest would Blake's \$1000 investment need to earn for it to have the same future value as Adam's investment?

EXAMPLE 3 Determining the interest rate of a regular payment investment

Jeremiah deposits $750 into an investment account at the end of every 3 months. Interest is compounded quarterly, the term is 3 years, and the future value is $10 059.07. What annual rate of interest does Jeremiah's investment earn?

Jeremiah's Solution



I drew a timeline to organize the given information and the information I needed so that I could determine the annual interest rate.

The regular payment amount is $750.
The payment frequency is 4 times per year.
The number of payments is 3(4) or 12.
The payments are made at the end of each payment period.
The annual interest rate is unknown.
The compounding frequency is 4 times per year.

I entered these values into the financial application on my graphing calculator and then solved for the annual interest rate.

The future value is $10 059.07.

Annual interest rate $= 0.080...$

The annual interest rate on Jeremiah's investment is 8.00%.

Your Turn

Predict whether the interest rate would be greater or less than 8% in each situation below, assuming the term remains 3 years, and the future value remains $10 059.07. Explain and then verify your prediction.

a) Jeremiah made payments of $800 every 3 months.
b) Jeremiah made payments of $750 every 6 months, and interest was compounded semi-annually.

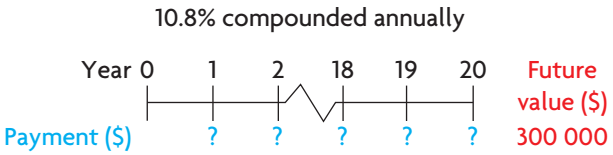
EXAMPLE 4 Determining the regular payment amount of an investment

Celia wants to have $300 000 in 20 years so that she can retire. Celia has found a trust account that earns a fixed rate of 10.8%, compounded annually.

a) What regular payments must Celia make at the end of each year to meet her goal of $300 000?
b) How much interest will she earn over the 20 years?

Liv's Solution

a)



I drew a timeline to organize the given information and the information I needed so that I could determine the payment amount.

The regular payment amount is unknown.
The payment frequency is 1 time per year.
The number of payments is 20.
The payments are made at the end of each payment period.
The annual interest rate is 10.8%.
The compounding frequency is 1 time per year.
The future value is $300 000.

I entered these values into the financial application on my graphing calculator and then solved for the payment amount.

Regular payment amount $= 4781.089\ldots$

Celia must make annual payments of $4781.09.

b) $I = 300\,000 - (20)(4781.089...)$
$I = 204\,378.20$

To determine the total interest earned, I subtracted the total of the 20 payments from the future value.

Celia will earn $204 378.20 in interest on 20 regular payments of $4781.09.

Your Turn

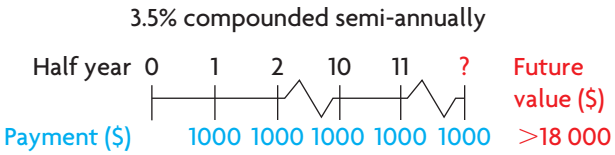
Predict whether Celia's payment amount would be greater than or less than $4781.09 in each situation. Explain and then verify your prediction.

a) Celia wants a future value of $500 000.
b) The payment frequency is every 6 months for 20 years (assume compounding is also every 6 months).
c) The interest rate is 7.8%, compounded annually.
d) Celia wants to have $300 000 in 10 years.

EXAMPLE 5 Determining the term of a regular payment investment

On Luis's 20th birthday, he started making regular $1000 payments into an investment account at the end of every 6 months. He wants to save for a down payment on a home. His investment earns 3.5%, compounded semi-annually. At what age will he have more than $18 000?

Greg's Solution



I drew a timeline to organize the given information and the information I needed. I could see that I had to figure out the number of semi-annual compounding periods in order to determine the number of years.

The regular payment amount is $1000.
The payment frequency is 2 times per year.
The number of payments is unknown.
The payments are made at the end of each payment period.
The annual interest rate is 3.5%.
The compounding frequency is 2 times per year.
The future value is $18 000.

I entered these values into the financial application on my graphing calculator and then solved for the number of payments.

Number of payments = 15.784...

The number of payments must be a whole number. Since Luis needs more than $18 000, I rounded up to 16 payments.

16 payments will result in more than $18 000.
16 payments made semi-annually is 8 years.

Luis will be 28 by the time his investment is worth more than $18 000.

Your Turn

What payments would Luis have to make if he wanted exactly $18 000 in 8 years?

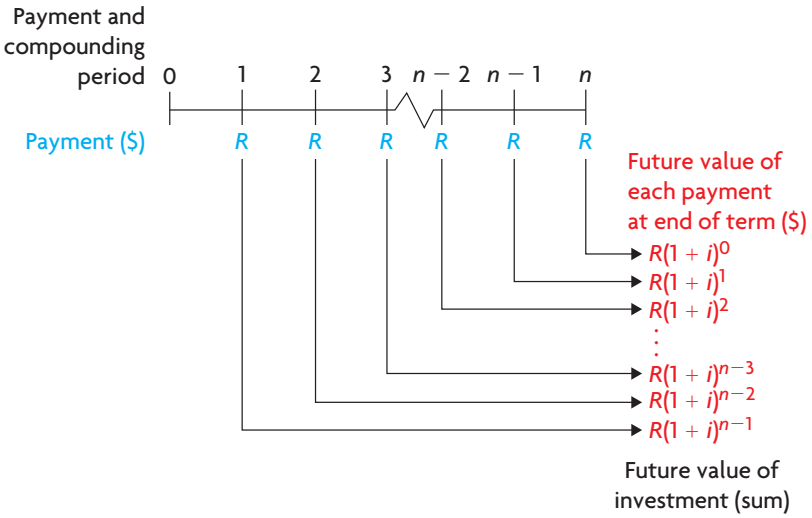
In Summary

Key Ideas

- For an investment that involves a series of equal deposits or payments made at regular intervals, the future value is the sum of all the regular payments plus the accumulated interest.
- The future value of an investment involving regular payments can be found by determining the sum of all the future values of each regular payment:

  $A = R(1 + i)^0 + R(1 + i)^1 + R(1 + i)^2 + R(1 + i)^3 + R(1 + i)^{n-1}$

  where A is the amount, or future value of the investment;
  R is the regular payment;
  i is the interest rate per compounding period, expressed as a decimal; and
  n is the number of compounding periods.



- Problems that involve the future value of an investment with regular payments can be solved using spreadsheet software or using the financial application on a graphing calculator or spreadsheet.

Need to Know

- The future value of a single deposit has a greater future value than a series of regular payments of the same total amount.
- Small deposits over a long term can have a greater future value than large deposits over a short term because there is more time for compound interest to be earned.

CHECK Your Understanding

1. Determine the future value of each investment.

	Regular Payment ($)	Interest Rate (%)	Compounding and Payment Frequency	Term (years)
a)	200	4.8	monthly	50
b)	1750	5.6	semi-annually	20
c)	50	8.4	quarterly	40
d)	5500	6.5	semi-annually	12

2. Determine the unknown values.

	Regular Payment ($)	Interest Rate (%)	Compounding and Payment Frequency	Term (years)	Future Value ($)
a)	100	?	monthly	6	7800.61
b)	?	3.50	semi-annually	7	3927.38
c)	20 000	4.75	quarterly	?	1 080 978.04

3. Darlene has invested $350 at the end of each month, at 7.2% compounded monthly, for 18 years. What is the investment's future value? How much interest has she earned?

4. Predict which investment will earn more interest. Explain and then verify your prediction.

A. $5000 invested at 6%, compounded annually, for 5 years

B. $1000 invested every year at 6%, compounded annually, for 5 years

PRACTISING

5. Fraser, who is 16 years old, wants to buy a car when he is 21. He deposits $600 every 3 months, from his part-time job, in a savings account that earns 6.8%, compounded quarterly. How much money will he have to buy his car when he is 21? How much interest will he have earned?

6. Zoey deposited the same amount of money at the end of each month for 2 years in a savings account that earned 6% interest, compounded monthly. She ended up with $5000. How much did Zoey deposit each month?

7. a) Jayne plans to retire in 35 years, when she is 55, and hopes to have $1 000 000 saved. For each investment option below, how much does she need to invest at the end of each month to reach her goal?

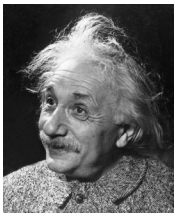
i) 14.6% compounded monthly

ii) 6.9% compounded monthly

b) Compare the rates of return for options i) and ii). Which option should she choose?

Math in Action

Wonder of the World?



Einstein allegedly claimed that "the most powerful force in the universe is compound interest." Certainly, banks and other financial institutions use compound interest as a cornerstone for their success in the business of making money. How do they do this? In simple terms, they earn the difference between what they charge their customers to borrow money and what they pay in interest to investors.

- Research the interest rate for a 5-year GIC and the interest rate for a 5-year loan offered by one or more financial institutions.
- Determine the difference between the amount of interest each institution would receive on a $50 000 loan and the amount of interest it would pay on a $50 000 investment.
- Make a conjecture about the difference between loan interest rates and investment interest rates, and explain your thinking.

8. Aaron and Casey started investing at the same time. Aaron makes payments of $25 at the end of each month into an investment that earns 4.2%, compounded monthly. Casey made a single payment into an investment that earns 4.2%, compounded annually.

a) At the end of 5 years, what is the future value of Aaron's investment?

b) Casey's investment has the same future value as Aaron's in 5 years. How much principal did Casey invest?

c) Predict whose investment will be worth more at the end of 10 years. Explain and then verify your prediction.

9. What interest rate, compounded monthly, is required to make monthly payments of $500 grow to $35 000 in 5 years?

10. How long will it take for $1000 payments every 6 months to grow to more than $10 000 if the interest rate is 7.5%, compounded semi-annually?

11. Dee deposited $1000 at the end of each month into a 5-year investment that earned 4%, compounded monthly. Pete deposited half as much each month for twice as long, but at the same interest rate.

a) Determine the future value of each investment.

b) Explain why the future values are different even though they invested the same amount.

12. For 2 years, Trey deposited $600 at the end of every 3 months into an investment that earned 3.6%, compounded quarterly. Over the same 2 years, Sam deposited $2400 annually into an investment that earned 3.8%, compounded annually. Whose investment is worth more, and by how much?

13. Miguel wants to buy an entertainment system as a wedding gift for his sister. He estimates that when she marries a year from now, the system will cost $2499, plus 13% tax. Will Miguel have enough money to buy the system if he deposits $225 at the end of each month into an account that earns 3.6%, compounded monthly? Explain.

14. Both Jill and Vaughn set up a 30-year investment and want to have $250 000 at the end of the term. Jill's bank pays a rate of 7.4%, compounded monthly. Vaughn is investing through the company he works for, at a rate of 11.6%, compounded monthly.

a) How much more does Jill need to invest than Vaughn over the 30 years?

b) Vaughn decides to make the same payments at the end of each month as Jill. How much will he have at the end of the 30 years?

15. Tim has found his dream sailboat in Victoria. It is selling for $120 000. He intends to sell his current sailboat in 2 years for $50 000. During those 2 years, Tim is going to put $300 at the end of each week into an investment account that earns 10.5%, compounded weekly. Will he have enough to buy his dream sailboat? Explain.

16. Andrew had two investment options:

A. A one-time deposit of $1200, which will earn 6%, compounded monthly, for 10 years

B. Deposits of $10 at the end of each month, which will earn 6%, compounded monthly, for 10 years

a) How are the investments the same? How are they different?

b) On the same grid, graph both investments. Compare their values over time. What do you notice?

Closing

17. Quinn is a server at a restaurant. He plans to deposit what he collects in tips each month for 5 years into an account that earns 5%, compounded monthly. He wonders how much he will be able to save altogether. Could you solve this as an investment involving regular payments? Explain.

Extending

18. Farah is a filmmaker. She paid $5000 for a computer that she can use to edit videos. She plans to sell this computer and buy a new one, with upgraded hardware, for $5000 in 2 years. In 2 years, her current computer will be worth about 25% of its current value. Farah started saving for her next computer by investing at the end of each month in an account that earns 3.6%/a, compounded monthly. How much should each payment be, so that she can pay cash for her next computer in 2 years?

19. When Blythe's adopted son was 6 months old, Blythe started depositing $50 at the end of each 6 months into an account that earned 5%, compounded semi-annually. When her son turned 20, she stopped making deposits but kept the money in the account. On her son's 30th birthday, she signed the account over to her son to help cover the start-up costs of his new business. How much did Blythe's son receive?

20. Pat wants his next vehicle to be environmentally friendly. He predicts that the vehicle will cost $46 000, plus 5% tax. He plans to invest $550 at the end of each month for the next 5 years into an investment that earns 9.8%, compounded monthly. If he sells his old car for $4000, will he have the purchase price of the new car? Explain.

1.6 Solving Investment Portfolio Problems

YOU WILL NEED

- spreadsheet software
- financial application on a graphing calculator

EXPLORE...

- Describe three different investments that would result in $30 000 in 10 years.

portfolio

One or more investments held by an individual investor or by a financial organization.

GOAL

Analyze, compare, and design investment portfolios that meet specific financial goals.

INVESTIGATE the Math

André is 17 years old and in Grade 12. He plans to start a 4-year history degree next year, focusing on the Métis of Western Canada. His goal is to be a curator at Le Musée de Saint-Boniface in Winnipeg. He has the following investments in his post-secondary education savings **portfolio**:

- Starting when he was born, his parents deposited $100 at the end of each month into a savings account, earning an average annual interest rate of 3%, compounded monthly.
- On his eighth birthday, his grandparents bought him a 10-year $5000 GIC that earned 4%, compounded annually.

André plans to redeem both investments now and combine them into one investment account that earns 4.2%, compounded quarterly, for one year until he starts school.

He expects to withdraw money from his investment account each year to cover major expenses, such as tuition, rent, and books. He also plans to work part-time while at university and during the summers.

Le Musée de Saint-Boniface is housed in Winnipeg's oldest building. The museum's collection includes items of cultural significance to both Métis and Francophone communities of Manitoba.

? Will André have enough money to pay for 4 years of university?

A. How much will André's parents' investment be worth when he redeems it?

B. How much will his grandparents' investment be worth when he redeems it?

C. How much will André's new investment account be worth when he starts school?

D. Suppose that André withdraws a lump sum from his investment account at the beginning of first year to cover his major first-year expenses. His summer job and part-time work during the year will pay for the rest of his first-year expenses. When he starts second year, his investment account has a value of $27 446.17. How much was his withdrawal at the beginning of first year?

E. Suppose that André withdraws the same amount for his second-year expenses. How much will he have in the account when he starts third year?

F. Suppose that André withdraws the same amount for his third-year expenses. Will he have enough left to withdraw the same amount again for his fourth-year expenses?

G. Suppose that André decides not to work part-time during fourth year because he wants to focus on his studies. What interest rate, compounded quarterly, would his account need to earn during his third year so that he can withdraw $11 500 for his fourth year?

Reflecting

H. When solving this problem, you determined

- the future value of a single investment,
- the future value of an investment involving regular payments,
- the present value of an investment, and
- the interest rate required for a single investment to grow to a certain future value.

Identify where you determined each value.

APPLY the Math

EXAMPLE 1 Determining the future value and doubling time of an investment portfolio

Phyllis started to build an investment portfolio for her retirement.

- She purchased a $500 Canada Savings Bond (CSB) at the end of each year for 10 years. The first five CSBs earned a fixed rate of 4.2%, compounded annually. The next five CSBs earned a fixed rate of 4.6% compounded annually.
- Three years ago, she also purchased a $4000 GIC that earned 6%, compounded monthly.

a) What was the value of Phyllis's portfolio 10 years after she started to invest?

b) Phyllis found a savings account that earned 4.9%, compounded semi-annually. She redeemed her portfolio and invested all the money in the savings account. About how long will it take her to double her money?

Sal's Solution

a)

	A	B	C	D	E
1	**Year**	***P* ($)**	***i***	***n***	***A* ($)**
2	**1**	500	0.042	9	724.0682
3	**2**	500	0.042	8	694.8831
4	**3**	500	0.042	7	666.8744
5	**4**	500	0.042	6	639.9946
6	**5**	500	0.042	5	614.1983
7	**6**	500	0.046	4	598.5449
8	**7**	500	0.046	3	572.2227
9	**8**	500	0.046	2	547.058
10	**9**	500	0.046	1	523
11	**10**	500	0.046	0	500
12				**Sum:**	6080.85

To figure out the value of the CSB part of the portfolio, I knew that I had to determine the future value of 10 bonds with two different interest rates. Then I had to determine the sum.

I decided that a spreadsheet would make the calculations easier.

To create the spreadsheet formula for the future values in column E, I used the following formula:

$$A = P(1 + i)^n$$

Since the bonds were purchased at the end of each year, the last bond would not earn any interest.

The CSB part of the portfolio has a value of $6080.85.

$6080.85 seemed reasonable because ten $500 CSBs would be $5000, not counting interest.

$4000 at 6% compounded monthly for 3 years:

$$A = P(1 + i)^n$$

$$A = 4000\left(1 + \frac{0.06}{12}\right)^{3(12)}$$

$$A = 4786.722...$$

Next, I determined the value of the GIC she purchased 3 years ago using the compound interest formula.

The GIC part of the portfolio has a value of $4786.72.

Portfolio value $= 6080.85 + 4786.72$
Portfolio value $= 10\ 867.57$
Phyllis's portfolio is worth $10 867.57.

I added the values of the CSBs and GIC to determine the total value of the portfolio.

b) Use the Rule of 72 to estimate the doubling time.

$$\frac{72}{4.9} = 14.694...$$

The portfolio will take about 14.5 years to double.

I rounded the estimate down because I knew that 14.694... was a bit high, due to semi-annual compounding.

Your Turn

Suppose that $21 735.14 was the future value of a single investment of $5000, made 31 years ago. What interest rate, compounded semi-annually, would the investment have had to earn?

EXAMPLE 2 Designing and adjusting an investment portfolio to meet a financial goal

John is an avid sailor and dreams about competing in the Olympics. He wants to buy his own Laser sailboat in 6 years, but, in the meantime, he sails on a friend's boat. The cost of a new Laser is about $9660, including taxes. John won $2500 in his most recent race and can save $50 a month from his part-time job.

a) What recommendations for a portfolio of two different investments would you make, based on available investments and interest rates? Explain.

b) Will the portfolio support the cost of the new Laser in 6 years?

c) If not, suggest a solution for John.

Lasers are small boats with only one sail. Of the classes of boats that compete at the Olympics, Laser is the least expensive.

Kyla's Solution

a) After researching available investments and interest rates, I recommend that John

- uses the $2500 cash to buy a 6-year GIC that earns 5.1%, compounded semi-annually, and
- deposits his monthly savings of $50 in a high-interest savings account that earns 4.3%, compounded monthly.

I wanted the cash to start earning interest right away at a good interest rate for 6 years. John also needs an account with a high interest rate, where he can deposit his monthly savings.

b) $2500 GIC that earns 5.1%, compounded semi-annually, for 6 years:

The present value is $2500.
The annual interest rate is 5.1%.
The compounding frequency is 2 times per year.
The number of years is 6.
The future value is unknown.

Future value = 3381.956...

I used the financial application on my calculator and entered these values to determine the future value of the GIC.

Deposits of $50 at the end of each month, earning 4.3%, compounded monthly, for 6 years:

The regular payment amount is $50.
The payment frequency is 12 times per year.
The number of payments is 6(12) or 72.
The payments are made at the end of each month.
The annual interest rate is 4.3%.
The compounding frequency is 12 times per year.
The future value is unknown.

Future value = 4098.726...

I entered these values into the financial application on my calculator to determine the future value of the savings account.

Total value = 3381.956... + 4098.726...
Total value = 7480.682...

John will have $7480.68 in 6 years, which is not enough.

c) $9660 - 7480.68 = 2179.32$
He needs $2179.32 more.

$$\frac{2179.32}{72} = 30.268...$$

John needs to save about $30 more each month.

I divided the additional amount he needs by the number of months in 6 years to estimate how much more he will need to save each month.

If he saves $30 more each month, he will have almost enough money, without counting interest. So, $30 per month will be enough.

$4098.73 + 2179.32 = 6278.05$
He needs to have a future value of $6278.05 in his savings account.
A regular payment amount of $76.60, or $26.60 more each month, will give him enough money in 6 years to buy the Laser.

I used the financial application on my calculator and tried different regular payment amounts until I got a future value of $6278.05.

I tried $75, $76, $77, $76.50, and finally $76.60.

Your Turn

a) What would John's monthly payments have to be if he wanted to buy the Laser in 5 years?
b) Suppose that the price of the Laser increases by 15% over the 6 years. How much would his monthly deposits have to be if he still wanted to buy the Laser in 6 years?

EXAMPLE 3 Comparing the rates of return of two investment portfolios

Jason and Malique are each hoping to buy a house in 10 years. They want their money to grow so they can make a substantial down payment.

Jason's portfolio:

- A 10-year $2000 GIC that earns 4.2%, compounded semi-annually
- A savings account that earns 1.8%, compounded weekly, where he saves $55 every week
- A 5-year $4000 bond that earns 3.9%, compounded quarterly, which he will reinvest in another bond at an interest rate of 4.1%

Malique's portfolio:

- A tax-free savings account (TFSA) that earns 2.2%, compounded monthly, and has a current balance of $5600
- The purchase, at the end of each year, of a 10-year $500 CSB that earns 3.6%, compounded annually
- A savings account that earns 1.6%, compounded monthly, where she saves $200 every month

In 10 years, whose portfolio will have the greater rate of return on investment?

Shelly's Solution

What Jason invested:

Total investment $= 2000 + 55(52)(10) + 4000$
Total investment $= 34\ 600$

Jason invested $34 600.

What Malique invested:

Total investment $= 5600 + 10(500) + 200(12)(10)$
Total investment $= 34\ 600$

Malique invested $34 600.

The rate of return for an investment portfolio, at any point in time, is the ratio of how much it has gained in value to how much was originally invested.

I began by determining how much was invested by Jason and Malique, including the principal amounts of the single payment investments and the total of the regular payments.

Jason's portfolio:

The GIC has a future value of $3030.71.
The savings account has a future value of $31 329.72.
The bond has a value of $4856.65 after 5 years and, after reinvesting, a future value of $5955.45.

Portfolio value $= 3030.71 + 31\ 329.72 + 5955.45$
Portfolio value $= 40\ 315.88$

Jason's portfolio has a future value of $40 315.88.

$$\text{Rate of return} = \frac{40\ 315.88 - 34\ 600}{34\ 600}$$

Rate of return $= 0.165...$

Jason's portfolio will have a rate of return of about 17%.

I used the financial application on my graphing calculator to determine the future value of each investment portfolio:

- For the single payment investments, I entered the term (in years), the present value, the annual interest rate, and the compounding frequency.
- For the investments involving regular payments, I entered the number of payments, the regular payment amount, the payment frequency, the time in the compounding period when the payments are made (end or beginning), the annual interest rate, and the compounding frequency.

To determine the rate of return, I subtracted the amount invested from the future value and then divided by the amount invested.

Malique's portfolio:

The TFSA has a future value of $6976.62.
The CSBs have a future value of $5892.88.
The savings account has a future value of $26 007.87.

Portfolio value $= 6976.62 + 5892.88 + 26\ 007.87$
Portfolio value $= 38\ 877.37$

Malique's portfolio has a future value of $38 877.37.

$$\text{Rate of return} = \frac{38\ 877.37 - 34\ 600}{34\ 600}$$

Rate of return $= 0.123...$

Malique's portfolio will have a rate of return of about 12%.

Jason's portfolio will have a higher rate of return than Malique's portfolio.

Your Turn

What recommendations would you make to help Malique increase her rate of return?

In Summary

Key Ideas

- Rate of return is a useful measure for comparing investment portfolios.
- An investment portfolio can be built from different types of investments, such as single payment investments (for example, CSBs and GICs) and investments involving regular payments. Some of these investments, such as CSBs, lock in money for specified periods of time, thus limiting access to the money, but offer higher interest rates. Other investments, such as savings accounts, are accessible at any time but offer lower interest rates. Investments that involve greater principal amounts invested or greater regular payment amounts when contracted tend to offer higher interest rates.
- The factors that contribute to a larger return on an investment are time, interest rate, and compounding frequency. The longer that a sum of money is able to earn interest at a higher rate compounded more often, the more interest will be earned. For investments involving regular payments, the payment frequency is also a factor.

Need to Know

- Financial applications on calculators or spreadsheets and online financial tools at banking websites are valuable tools for analyzing and comparing investment portfolios.

CHECK Your Understanding

1. Stan plays in a band. Next year, he wants to have enough money to buy a new guitar. The new guitar costs $1750, including taxes and shipping. Stan works part time and can afford to save $15 every week. As well, he has $300 left from his summer job. He needs an investment portfolio so that he can save money to buy the guitar in a year.

a) Why might Stan include a GIC and a high-interest savings account in his portfolio?

b) If the GIC earns 5%, compounded annually, and the savings account earns 2.9%, compounded weekly, will he have enough money in a year? If not, how much does he have to save each week?

2. In 2 years, Robin and Leslie are getting married. They anticipate that the wedding will cost about $10 000.

- They plan to save $50 a week in an investment account that earns 4.1%, compounded weekly.
- Their parents have offered to contribute and would like to know how much to give them. They plan to invest their parents' contribution in a CSB that earns 6%, compounded annually.

How much should their parents give them?

PRACTISING

3. Hugh has created the following investment portfolio:

- At the end of each year, for the past 10 years, he has purchased a $1000 CSB, with an average annual interest rate of 3.4%, compounded annually.
- He has a trust account that was set up when he was born, 42 years ago, with a single deposit of $3000. The trust fund earns an average annual interest rate of 4.3%, compounded quarterly.
- He has a $10 000 GIC, with a 10-year term, that he purchased 10 years ago and earned 3.95%, compounded semi-annually.

Hugh intends to redeem everything and then invest all the money in a 5-year bond that earns 5.1%, compounded annually. How much will Hugh's bond be worth in 5 years?

4. Paula just turned 18 and is about to start a 3-year college program. She lives with her family, but she still needs about $2000 each year for expenses.

- Paula has been working part-time for the past 3 years and has deposited $50 each month into an investment account that earns 2.7%, compounded monthly.
- When she was born, her parents opened an RESP account that earns 3.2%, compounded monthly. Her parents have deposited $10 each month into this account.

a) How much money does Paula have when she starts first year?

b) Paula decides to redeem her investments when she starts first year, and she withdraws $2000 for her expenses. She then reinvests the rest of the money in a savings account that earns 3.5%, compounded daily. Will she have enough money for her expenses when she starts second year? Explain.

c) If Paula withdraws another $2000 for second year, will she have enough money for third year? If not, how much does she need to save over the summer between second and third year?

5. Gayla and Corey are both 35 years old but have been investing different amounts of money for different lengths of time.

- Gayla saved $15 each month for 12 years at an average annual interest rate of 3.6%, compounded monthly, until she was 25. Then she reinvested the entire amount at 3.8%, compounded monthly, for 10 years.
- Corey has been depositing $18 each month, for the past 10 years, into a savings account that earns an average annual interest rate of 3.8%, compounded monthly.

a) How much did each woman invest altogether?

b) Predict who has the greater savings. Explain your prediction, and then verify it.

6. Jayne's investment portfolio is described below.

- When Jayne was born, 40 years ago, her parents opened a trust account for her. They invested $500 at the end of each year into the trust account until she was 20. Since then, there have been no more deposits, but the account has continued to earn interest at an average annual rate of 5%, compounded annually.
- 10 years ago, Jayne purchased a 10-year $10 000 GIC that earned 4.4%, compounded semi-annually.
- 5 years ago, she started buying a 5-year $1000 CSB at the beginning of each year. The first two CSBs earned 4.7%, compounded annually; the next two CSBs earned 4.8%, compounded annually; and the last CSB earned 4.9%, compounded annually.

How much is Jayne's investment portfolio worth now? What is her rate of return?

7. When Ela was 20, he started to build an investment portfolio.

- He opened a savings account and invested $50 a month until he was 40 and earned an average annual rate of 2.7%, compounded monthly.
- When Ela was 40, he redeemed the savings account and invested the entire amount in a 10-year GIC that earned 4.2%, compounded monthly.
- At maturity, he reinvested the entire value of the GIC in another 10-year GIC that earned 4.3%, compounded monthly.
- When he was 40, he also purchased a 10-year $500 bond that earned 3.9%, compounded annually.
- He reinvested the money at the same interest rate when the bond matured.

a) What is the value of Ela's portfolio when he turns 60?

b) What is Ela's rate of return?

8. In 4 years, Jo wants to travel somewhere exotic. She estimates that she will need about $2000 for her flights and about $300 a week for 12 weeks of travel. She has savings of $1200 that she can invest immediately.

a) Suppose that she invests her savings in a 4-year GIC that earns 5%, compounded annually. Will she have enough money in 4 years?

b) If not, estimate how much she will need to save each week in a savings account that earns 2.3%, compounded weekly. Explain your estimate, and then determine the exact amount.

9. Derek was laid off, after 20 years of service. His severance pay was $18 638. Since Derek found another job immediately, he decided to invest his severance pay. Which of the two options would you advise him to choose for the next 10 years? Explain.

A. A 10-year $15 000 GIC at 4.1%, compounded annually, and a high-interest savings account at 3.9%, compounded weekly, for the remaining $3638

B. A high-interest savings account at 3.9%, compounded daily, for all the severance pay

Closing

10. What factors do you need to keep in mind when creating or evaluating an investment portfolio? Explain.

Extending

11. Betty is retiring and has $200 000 in savings. She wants to invest the $200 000 in an investment that will earn interest at 5%, compounded monthly. She wants to receive equal monthly payments from the account for the next 10 years, starting next month. At the end of the 10 years, the account will have a zero balance.

a) How much will her payment be each month?

b) How much would her monthly payment be if the interest rate was 7.5%, compounded monthly?

12. Stu's 75-year-old grandmother set up an investment portfolio when she was 20. She invested $150 every year and earned an average annual interest rate of 8.8%, compounded annually. When she was 70 years old, she redeemed the investment and invested all the money in an account that earned 4.8%, compounded monthly. While the investment earns interest, she receives a monthly allowance from the investment. How much will Stu's grandmother receive altogether in payments if the investment will have a zero balance at the end of 10 years?

1 Chapter Self-Test

1. Hal invested $40 000 at an interest rate of 6%, compounded annually. He wants to know how long it will take for the investment to double.
   a) Estimate the doubling time. Verify your answer.
   b) How long would it take for the investment to double if the interest was simple interest?

2. Val has $12 000 and wants it to grow to $50 000. She has narrowed the possibilities down to the following two investment options:
   A. 6% compounded semi-annually **B.** 5.1% compounded quarterly
   Which option should she choose? Why?

3. Warren started investing when he was 5 years old. He deposited $5 from his allowance at the end of every month into a savings account that earned 5.8%, compounded monthly. He did this until he was 25 years old.
   a) How much did he invest altogether? What is the current value of his investment at age 25? What is his rate of return?
   b) Suppose that he had invested the same total amount in regular equal monthly deposits, but had not started until he was 20. What would the current value of his investment be?
   c) Suppose that Warren had wanted his investment to have the same value as in part a) at age 25, but had started investing when he was 20. What would his monthly payments have been?

4. Both Alex and Jamie have an investment portfolio.
   a) What is the current value of each portfolio?
   b) Who has the greater rate of return? Explain.

Alex's portfolio:
- A 10-year $5000 GIC, purchased 9 years ago, that earns 2.6%, compounded annually
- A 5-year $2000 CSB, purchased 4 years ago, that earns 3.1%, compounded semi-annually
- A savings account at 1.4%, compounded weekly, into which he has been making weekly deposits of $15 for 5 years

Jamie's portfolio:
- A 10-year $3000 bond, purchased 9 years ago, that earns simple interest at 2.7%
- A 3-year $700 CSB, purchased 3 years ago, that earns 2.8%, compounded semi-annually
- A high-interest savings account, at 1.7% compounded monthly, into which she has been making monthly deposits of $100 for 6 years

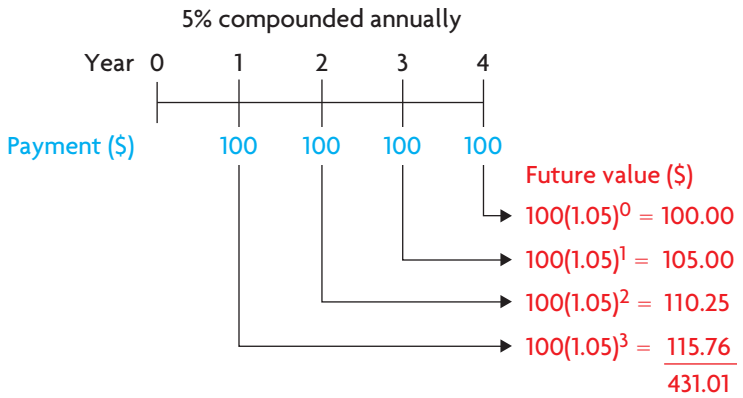
***WHAT** DO You Think Now?* Revisit **What Do You Think?** on page 5. How have your answers and explanations changed?

1 Chapter Review

FREQUENTLY ASKED Questions

Q: How does the future value of a single payment investment compare with the future value of an investment involving regular payments?

A: How they are alike:
Each payment in a regular payment investment is like a single payment investment. The future value of the regular payment investment is the sum of the future values of all the payments. For example, the timeline below shows an investment involving $100 payments made at the end of each year at 5% interest, compounded annually, for 4 years.



How they are different:
If the four $100 payments had been invested as a single payment, under the same conditions, the future value would be greater than $431.01. This is because the entire amount invested in a single payment investment earns compound interest for the entire term. In contrast, if an investment involves regular payments, only the first payment earns interest for the full term. Each payment after that earns interest over less time.

Study Aid
- See Lesson 1.5, Examples 1 to 5.
- Try Chapter Review Questions 8 to 10.

Study **Aid**

- See Lesson 1.6, Examples 1 to 3.
- Try Chapter Review Questions 11 and 12.

Q: How can you evaluate or compare investment portfolios that are made up of multiple investments?

A: You can use their rates of return. The rate of return is the ratio of interest earned to the amount invested. It is particularly useful when comparing investments in which the principal or the term is different. For example, an investor is considering the following two portfolio options. Portfolio B has a greater amount invested but for a shorter term than portfolio A. Which portfolio will have the greater rate of return?

Portfolio A $25 000 invested for 4 years: i) $20 000 in a tax-free savings account, earning an average annual interest rate of 1.3%, compounded annually each year for 4 years ii) A $5000 Canada Savings Bond at 5%, compounded annually	***Determining rate of return*** i) Future value: 21 060.46 Interest earned: 21 060.46 − 20 000 or 1060.46 ii) Future value: 6077.53 Interest earned: 6077.53 − 5000 or 1077.53 Total interest earned: 1060.46 + 1077.53 or 2137.99 Total amount invested: 25 000 Rate of return: $\frac{2137.99}{25\,000}$ or 8.55%
Portfolio B $30 000 invested for 3 years: i) $10 000 earning an average annual interest rate of 3%, compounded monthly ii) A $20 000 GIC earning 2.5% simple interest	***Determining rate of return*** i) Future value: 10 940.51 Interest earned: 10 940.51 − 10 000 or 940.51 ii) Future value: 21 500 Interest earned: 21 500 − 20 000 or 1500 Total interest earned: 940.51 + 1500 or 2440.51 Total amount invested: 30 000 Rate of return: $\frac{2440.51}{30\,000}$ or 8.14%

Portfolio A has a greater rate of return than portfolio B.

PRACTISING

Lesson 1.1

1. When Trish was 12, her grandmother bought her a $1500 CSB that earned simple interest.

a) When Trish turned 18, her CSB was worth $2850. At what rate did the CSB earn interest?

b) If the interest was paid annually and Trish redeemed the CSB when she was 18 and a half years old, how much would she get? Explain.

2. Steve is celebrating his 18th birthday.

- On his 5th birthday, his grandmother bought him a $10 000 GIC that earns 6.3% simple interest.
- On the same birthday, his grandfather bought him a $7000 CSB that earns 11.4% simple interest.

a) What is the value of each investment now?

b) Graph both investments on the same grid to show how the values of the investments change over time.

c) What conclusion can you draw from comparing the graphs?

Lesson 1.2

3. Examine these two investments, and then answer the questions below.

- Sonia invested in a $2000 GIC that earns 6.2% simple interest, paid annually, for 5 years.
- Trent bought a $2000 GIC that earns 5.3%, compounded monthly, for a 5-year term.

a) Predict which investment will have the greater rate of return. Explain.

b) Verify your prediction.

c) Explain the difference in the interest earned on the two investments.

Lesson 1.3

4. James and Johnny received equal inheritances of $2000, which they invested for 5 years at 7.4%. James's account compounded semi-annually, and Johnny's account compounded weekly.

a) Predict who will earn more interest. Verify your answer.

b) Compare their rates of return.

5. Kyle invested $900 in an account that earned interest, compounded daily. After 1 year, her investment was worth $1000. What was the annual rate of interest?

Lesson 1.4

6. Phil and his daughter Lina opened accounts at different times. Each account earned 6.5%, compounded semi-annually.

- Phil kept his account for 18 years and now has $125 000 in the account.
- Lina kept her account for 36 years and now has $125 000 in the account.

a) Who invested the greater principal? How much more did he or she invest?

b) If Lina had invested the same principal as Phil, what would be the future value of her account after 36 years?

7. Mel and his friend Mike each made an investment.

- Mel invested, 10 years ago, in a bond that matures this year. The bond currently has a value of $13 140 and has been earning 6.5%, compounded annually.
- Mike invested in a 10-year GIC that also matures this year. Mike's investment is currently worth $13 009 and has been earning 9.6%, compounded monthly.

a) Who invested the greater principal 10 years ago?

b) Graph both investments on the same grid.

c) Make a prediction about what will happen to the future values of the two investments. Explain.

Lesson 1.5

8. Mary decided to invest $800 per month for the next 6 years. She plans to start her own business making lunches for the elementary schools in her area, but she needs to save enough money for her start-up costs.

- Bank A has offered her 12.2%, compounded monthly.
- Bank B has offered her 11.4%, compounded monthly.

If Mary chooses bank A, how much more money will she end up with?

9. Two brothers, Josh and Jeff, held investments that earned 6%, compounded annually. Both of them made regular payments into their investments until they were 65.

- Josh started making yearly payments of $1000 when he was 20.
- Jeff did not start until he was 50, but made annual deposits of $3000.

a) What is the future value of each investment?
b) How much did each man invest altogether?
c) How much interest did each man earn?
d) What annual deposit would Jeff have needed to make if he had wanted his investment to have the same future value as Josh's investment at age 65?

10. Drew and John are both planning to buy a new car in 5 years. Each of them started an investment to save the money they will need.

- Drew deposited $100 every 2 weeks and earned 4.8%, compounded biweekly.
- John deposited $217 every month and earned 4.8%, compounded monthly.

a) Why is it difficult to predict who will be able to purchase the more expensive car in 5 years?
b) Determine who will be able to purchase the more expensive car.

Lesson 1.6

11. When Chandra was 8 years old, an investment portfolio was started for her education.

- Her parents deposited $450 every 3 months into a savings account that earns 4.5%, compounded quarterly.
- Her grandparents invested $5000 in a trust account that earns an average annual interest rate of 6%, compounded annually.

a) Chandra plans to redeem her portfolio when she turns 18 to pay for university.
  i) What will be the value of her portfolio?
  ii) What was the portfolio's rate of return?
b) If she withdraws $10 000 each year for university and then invests what is left each time at 6.2%, compounded annually, will she have enough for 4 years?

12. **a)** Predict which portfolio will have the greater rate of return over 10 years. Explain your prediction.
b) Verify your prediction. Was your prediction correct?

Portfolio 1:

- A 10-year $25 000 GIC, purchased 10 years ago, that earns 8.7%, compounded annually
- A 10-year $10 000 CSB, purchased 10 years ago, that earns 6.4% simple interest
- An investment involving deposits of $2500 at the end of every year for 10 years, which earns 4.9%, compounded annually

Portfolio 2:

- A 10-year $25 000 GIC, purchased 10 years ago, that earns 8.7%, compounded monthly
- A 10-year $10 000 CSB, purchased 10 years ago, that earns 6.4%, compounded annually
- An investment involving deposits of $1250 at the end of every 6 months for 10 years, which earns 4.9%, compounded semi-annually

1 Chapter Task

Investing to Travel the World

Helen and Noelle want to travel around the world together in 3 years. They have decided to make a financial commitment toward their goal by setting up an investment portfolio. They estimate that they will need about $23 280, in total, to travel for 3 months.

Expense	Estimate
airline tickets, $3000 each	$7 000
rail passes, $2000 each	$5 000
hostels, $15/day each for 92 days	$2 760
meals, $30/day each for 92 days	$5 520
incidentals	$3 000
Total	**$23 280**

Helen and Noelle currently have a total of $6000, which could be invested now. They are both working full-time, so they could also save money each month.

Their portfolio consists of

- a 3-year GIC that earns an average annual interest rate of 4.5%, compounded annually, and
- regular payments into an investment savings account with an average interest rate of 3.9% compounded monthly.

? Will Helen and Noelle meet their financial goal?

A. Does their portfolio seem reasonable?

B. Helen and Noelle wonder if they would have enough money with just the GIC. Estimate how long it will take for $6000 to grow to about $23 280 at 4.5%, compounded annually.

C. How much do Helen and Noelle need to save each month to achieve their goal? Explain.

D. Suppose that, after 3 years, the cost of the trip turns out to be 8% greater than their estimate. How much longer will they have to wait until they have enough money? Explain.

Task *Checklist*

- ✔ Did you provide your reasoning?
- ✔ Did you check that your results are reasonable?
- ✔ Is your advice clear and easy to understand?
- ✔ Did you use correct mathematical terms?

1 Project Connection

Creating an Action Plan

Deadlines are part of life. Completing projects on time is just as important in the workplace as it is in school. So, how can you avoid having to rush through all the stages of your research project at the last minute? One way is to use a strategy called backward planning: develop a formal action plan, and create a timeline based on this action plan.

A major research project must successfully pass through several stages. On the next page is an outline for an action plan, with a list of these stages. Completing this action plan will help you organize your time and give you goals and deadlines you can manage. The times that are suggested for each stage are only a guide, with one day equivalent to any regular day in your life. Adjust the time you will spend on each stage to match the scope of your project. For example, a project based on primary data (data that you collect) will usually require more time than a project based on secondary data (data that other people have collected and published). You will also need to consider your personal situation—the issues that are affecting you and may interfere with completion of the project.

Issues Affecting Project Completion

Consider the issues that may interfere with completion of the project in a time-efficient manner. For example:

- part-time job
- after-school sports and activities
- regular homework
- assignments for other courses
- tests in other courses
- driving school
- time you spend with friends
- school dances and parties
- family commitments
- access to research sources and technology

What other issues can you add to this list?

Your Turn

A. Take some time to complete an action plan for your project. Start by deciding on the probable length of time for each stage. Do not forget to include buffer space in your action plan. Buffer space is not a stage, but it is important. If something goes wrong (for example, if you are unable to gather appropriate data for your topic and must select a new topic), having that buffer space in your action plan may allow you to finish your project on time, without making extraordinary efforts.

1. **Select the topic you would like to explore.**
   Suggested time: 1 to 3 days
   Your probable time:
   Finish date:

2. **Create the research question that you would like to answer.**
   Suggested time: 1 to 3 days
   Your probable time:
   Finish date:

3. **Collect the data.**
   Suggested time: 5 to 10 days
   Your probable time:
   Finish date:

   Buffer space
   Suggested time: 3 to 7 days
   Your probable time:
   Finish date:

4. **Analyze the data.**
   Suggested time: 5 to 10 days
   Your probable time:
   Finish date:

5. **Create an outline for your presentation.**
   Suggested time: 2 to 4 days
   Your probable time:
   Finish date:

6. **Prepare a first draft.**
   Suggested time: 3 to 10 days
   Your probable time:
   Finish date:

7. **Revise, edit, and proofread.**
   Suggested time: 3 to 5 days
   Your probable time:
   Finish date:

8. **Prepare and practise your presentation.**
   Suggested time: 3 to 5 days
   Your probable time:
   Finish date:

B. Use a calendar and your probable times for each stage to work backwards from the presentation date to create a schedule you can follow. This will ensure that you will be able to complete all the stages of your project in the time available. In your schedule, include regular conferences with your teacher (5 to 10 min) to discuss your progress.