3.1 Types of Sets and Set Notation

YOU WILL NEED
- compass

EXPLORE...
- What categories can you use to organize your clothes?

set
A collection of distinguishable objects; for example, the set of whole numbers is $W = \{0, 1, 2, 3, \ldots\}$.

element
An object in a set; for example, 3 is an element of D, the set of digits.

universal set
A set of all the elements under consideration for a particular context (also called the sample space); for example, the universal set of digits is $D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

subset
A set whose elements all belong to another set; for example, the set of odd digits, $O = \{1, 3, 5, 7, 9\}$, is a subset of D, the set of digits. In set notation, this relationship is written as:
$O \subset D$

complement
All the elements of a universal set that do not belong to a subset of it; for example, $O' = \{0, 2, 4, 6, 8\}$ is the complement of $O = \{1, 3, 5, 7, 9\}$, a subset of the universal set of digits, D. The complement is denoted with a prime sign, O'.

GOAL
Understand sets and set notation.

INVESTIGATE the Math

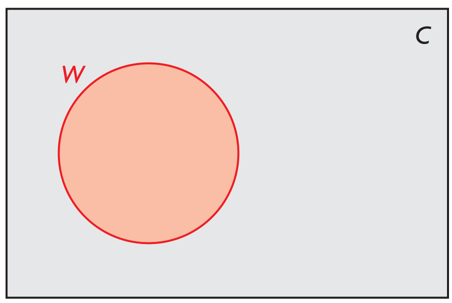
Jasmine is studying the provinces and territories of Canada. She has decided to categorize the provinces and territories using **sets**.

? How can Jasmine use sets to categorize Canada's regions?

A. List the **elements** of the **universal set** of Canadian provinces and territories, C.

B. One **subset** of C is the set of Western provinces and territories, W. Write W in set notation.

C. The Venn diagram to the right represents the universal set, C. The circle in the Venn diagram represents the subset W.



The **complement** of W is the set W'.

i) Describe what W' contains.
ii) Write W' in set notation.
iii) Explain what W' represents in the Venn diagram.

D. Jasmine wrote the set of Eastern provinces as follows:

$E = \{\text{NL, PE, NS, NB, QC, ON}\}$

Is E equal to W'? Explain.

E. List T, the set of territories in Canada. Is T a subset of C? Is it a subset of W, or a subset of W'? Explain using your Venn diagram.

F. Explain why you can represent the set of Canadian provinces south of Mexico by the **empty set**.

G. Consider sets C, W, W', and T. List a pair of **disjoint** sets. Is there more than one pair of disjoint sets?

H. Complete your Venn diagram by listing the elements of each subset in the appropriate circle.

empty set

A set with no elements; for example, the set of odd numbers divisible by 2 is the empty set.

The empty set is denoted by { } or $\varnothing$.

disjoint

Two or more sets having no elements in common; for example, the set of even numbers and the set of odd numbers are disjoint.

Reflecting

I. Why might you use a Venn diagram instead of a map to categorize the regions of Canada? Explain with an example.

J. **i)** What other sets could you use to sort the provinces and territories?
ii) Which of these new sets are subsets of the sets you used earlier?
iii) Which of these new sets are disjoint?

Communication | *Notation*

The following is a summary of notation introduced so far.

Sets are defined using brackets. For example, to define the universal set of the numbers 1, 2, and 3, list its elements:

$U = \{1, 2, 3\}$

To define the set A that has the numbers 1 and 2 as elements:

$A = \{1, 2\}$

All elements of A are also elements of U, so A is a subset of U:

$A \subset U$

The set A', the complement of A, can be defined as:

$A' = \{3\}$

To define the set B, a subset of U that contains the number 4:

$B = \{\ \}$ or $B = \varnothing$

$B \subset U$

APPLY the Math

EXAMPLE 1 Sorting numbers using set notation and a Venn diagram

a) Indicate the multiples of 5 and 10, from 1 to 500, using set notation. List any subsets.
b) Represent the sets and subsets in a Venn diagram.

Ramona's Solution

a) $S = \{1, 2, 3, ..., 498, 499, 500\}$

I defined S as the universal set of all natural numbers from 1 to 500.

$S = \{x \mid 1 \leq x \leq 500, x \in N\}$

There were too many numbers to list, so I wrote an expression for the set. $1 \leq x \leq 500$ means that x can be any number from 1 to 500.

$F = \{5, 10, 15, ..., 490, 495, 500\}$
$F = \{f \mid f = 5x, 1 \leq x \leq 100, x \in N\}$

I defined F as the set of multiples of 5 from 1 to 500. Since the greatest element of F is 500, I wrote the set as a multiple of 5 using values of x from 1 to 100.

$F \subset S$

F is a subset of S since all the elements of F are also elements of S.

$T = \{10, 20, 30, ..., 480, 490, 500\}$
$T = \{t \mid t = 10x, 1 \leq x \leq 50, x \in N\}$

I defined T as the set of all multiples of 10 from 1 to 500.

Since the greatest element of T is 500, I wrote it as a multiple of 10 using values of x from 1 to 50.

$T \subset F \subset S$

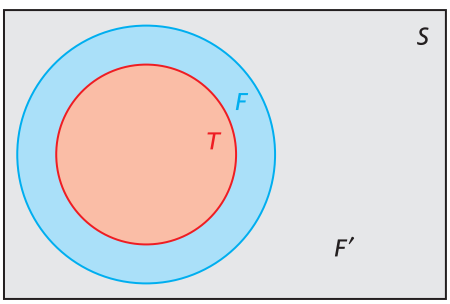
T is a subset of both F and S.

$F' = \{\text{non-multiples of 5 from 1 to 500}\}$

The complement of F, F', contains all the numbers that are not multiples of 5.

There are too many numbers to list, and I can't write "non-multiples of 5" using an algebraic expression, so I wrote a description.

b)



I showed the relationships among the sets in a Venn diagram.

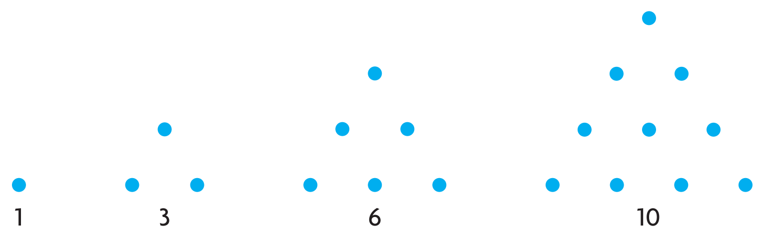
Since $T \subset F$, the circle that represents the subset T is inside the circle that represents the subset F.

Your Turn

Indicate the multiples of 4 and 12, from 1 to 240 inclusive, using set notation. List any subsets, and show the relationships among the sets and subsets in a Venn diagram.

EXAMPLE 2 Determining the number of elements in sets

A triangular number, such as 1, 3, 6, or 10, can be represented as a triangular array.



a) Determine a pattern you can use to determine any triangular number.
b) Determine how many natural numbers from 1 to 100 are
  i) even and triangular,
  ii) odd and triangular, and
  iii) not triangular.
c) How many numbers are triangular?

Simon's Solution

a) I examined the pattern of dots for the triangular numbers 1, 3, 6, and 10 to find a pattern.

1 — The first triangular number is 1.

3 $\quad 1 + 2$ — The second triangular number is the sum of the first two natural numbers.

6 $\quad 1 + 2 + 3$ — The third triangular number is the sum of the first three natural numbers.

10 $\quad 1 + 2 + 3 + 4$ — The fourth triangular number is the sum of the first four natural numbers.

The *n*th triangular number is the sum of the first *n* natural numbers. — Each triangular number is a sum of consecutive natural numbers, beginning at 1.

b) i) I used a spreadsheet to generate more triangular numbers.

	A	B
1	Natural Numbers	Triangle Numbers
2	1	1
3	2	3
4	3	**6**
5	4	**10**
6	5	15
7	6	21
8	7	**28**
9	8	**36**
10	9	45
11	10	55
12	11	**66**
13	12	**78**
14	13	91
15	14	105

I entered the natural numbers in column A.

I entered the first triangular number, 1, in cell B2.

To define the second triangular number, I entered the formula = B2 + A3 in cell B3.

I dragged the formula down until I reached a triangular number greater than 100.

I put all the even triangular numbers in boldface.

Communication | *Notation*

The phrase "from 1 to 5" means "from 1 to 5 inclusive."

In set notation, the number of elements of the set X is written as $n(X)$.

For example, if the set X is defined as the set of numbers from 1 to 5:

$X = \{1, 2, 3, 4, 5\}$
$n(X) = 5$

$U = \{$natural numbers from 1 to 100$\}$
$T = \{$triangular numbers from 1 to 100$\}$
$E = \{$even triangular numbers from 1 to 100$\}$

I defined the universal set, U, and subsets T and E.

$T = \{1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91\}$
$E = \{6, 10, 28, 36, 66, 78\}$

I listed the elements in T and E.

$n(T) = 13$
$n(E) = 6$
There are 13 triangular numbers from 1 to 100, and 6 of these numbers are even.

I determined the number of elements in each **finite set** by counting.

finite set
A set with a countable number of elements; for example, the set of even numbers less than 10, $E = \{2, 4, 6, 8\}$, is finite.

ii) $O = \{$odd triangular numbers from 1 to 100$\}$
$n(O) = n(T) - n(E)$
$n(O) = 13 - 6$
$n(O) = 7$

There are 7 odd triangular numbers from 1 to 100.

I defined the set of odd triangular numbers, O.

Since triangular numbers are either even or odd, I subtracted the number of even triangular numbers from the total number of triangular numbers.

iii) $n(U) = 100$
$n(T') = n(U) - n(T)$
$n(T') = 100 - 13$
$n(T') = 87$

There are 87 numbers from 1 to 100 that are not triangular.

The universal set contains 100 elements. These elements are either triangular numbers or they are not.

I subtracted to determine the number of elements in subset T', the set of non-triangular numbers.

c) There is an infinite number of natural numbers, so there must be an infinite number of triangular numbers. The set of triangular numbers is an **infinite set**.

The pattern for triangular numbers continues forever. I can start to count the triangular numbers, but it is impossible to count all of them.

infinite set

A set with an infinite number of elements; for example, the set of natural numbers, $N = \{1, 2, 3, \ldots\}$, is infinite.

Your Turn

Explain why Simon defined O as a separate subset, rather than using E'.

EXAMPLE 3 Describing the relationships between sets

Alden and Connie rescue homeless animals and advertise in the local newspaper to find homes for the animals. They are setting up a web page to help them advertise the animals that are available. They currently have dogs, cats, rabbits, ferrets, parrots, lovebirds, macaws, iguanas, and snakes.

a) Design a way to organize the animals on the web page. Represent your organization using a Venn diagram.
b) Name any disjoint sets.
c) Show which sets are subsets of one another using set notation.
d) Alden said that the set of fur-bearing animals could form one subset. Name another set of animals that is equal to this subset.

Connie's Solution

a) I defined the universal set, A.

A = {all the animals that are available}
W = {warm-blooded animals}
C = {cold-blooded animals}

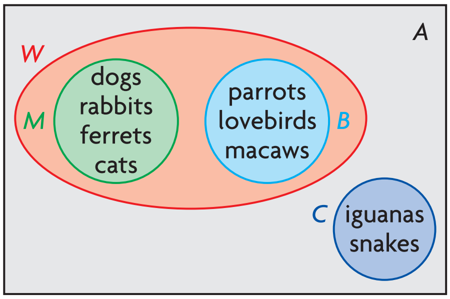
The animals are either warm-blooded or cold-blooded. I defined the subsets W and C for these two types of animals.

I listed the elements in each subset.

W = {dogs, cats, rabbits, ferrets, parrots, lovebirds, macaws}
C = {iguanas, snakes}

I decided to organize the subset of warm-blooded animals into two further subsets: mammals, M, and birds, B,

M = {dogs, cats, rabbits, ferrets}
B = {parrots, lovebirds, macaws}



I drew circles to represent subsets M (mammals), B (birds), and C (cold-blooded animals). Subsets M and B form the set W (warm-blooded animals), so I drew an oval around these two circles.

Then I drew a rectangle around all the shapes to represent the universal set, A (all available animals).

b) The disjoint sets are W and C, M and C, B and C, and M and B.

I looked for shapes that do not overlap. I knew that these sets do not contain common elements.

c) $M \subset W$, $B \subset W$
$M \subset A$, $B \subset A$, $C \subset A$, $W \subset A$

Sets M and B are inside W, so they are subsets of W. All the sets are subsets of the universal set, A.

d) $F = \{\text{fur-bearing animals}\}$
$F = \{\text{cats, dogs, ferrets, rabbits}\}$
$M = F$
Set M is equal to set F, the set of fur-bearing animals.

I defined set F. Then I listed the elements in F.

Set M contains the same elements as F, so these sets are equal.

It does not matter that the animals in M are listed in a different order than the animals in F.

Your Turn

How else might you categorize the animals into sets and subsets?

EXAMPLE 4 Solving a problem using a Venn diagram

Bilyana recorded the possible sums that can occur when you roll two four-sided dice in an outcome table.

	1	2	3	4
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8

a) Display the following sets in one Venn diagram:
- rolls that produce a sum less than 5
- rolls that produce a sum greater than 5

b) Record the number of elements in each set.

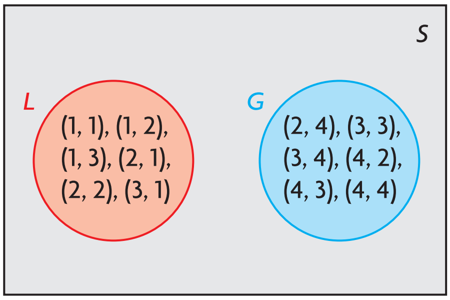
c) Determine a formula for the number of ways that a sum less than or greater than 5 can occur. Verify your formula.

Morgan's Solution

a) $S = \{\text{all possible sums}\}$
$L = \{\text{all sums less than 5}\}$
$G = \{\text{all sums greater than 5}\}$

I defined the universal set S and the subsets L and G.

I represented the relationship between the sums in a Venn diagram.



Since it is not possible for a sum to be less than 5 and greater than 5 at the same time, the subsets are disjoint. The events that describe the subsets are **mutually exclusive**.

mutually exclusive
Two or more events that cannot occur at the same time; for example, the Sun rising and the Sun setting are mutually exclusive events.

b) $n(S) = 16$
$n(L) = 6$
$n(G) = 6$

I counted the number of elements in each set using the outcome table.

c) The number of ways that these events can happen is the sum of the number of ways that each event can happen.

$n(L \text{ or } G) = n(L) + n(G)$
$n(L \text{ or } G) = 6 + 6$
$n(L \text{ or } G) = 12$

Events L and G are mutually exclusive. This is the sum of the number of elements in each set.

A sum that is greater than 5 or less than 5 can occur in 12 ways.

There are 4 ways that either sum cannot happen. Since the dice can fall in 16 ways, the sums that are greater than 5 or less than 5 can happen in 12 ways.

I verified my results using my table. There are 4 sums that are equal to 5 out of 16 possible sums.

Your Turn

Two six-sided dice are rolled.

a) Illustrate the following sets in one Venn diagram:
- rolls that produce a sum less than 6
- rolls that produce a sum greater than 6

b) Record the number of elements in each set.

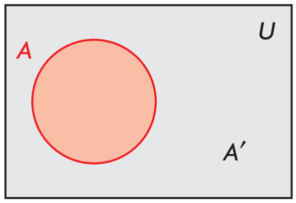
c) Determine a formula for the number of ways that a sum less than or greater than 6 can occur. Verify your formula.

In Summary

Key Ideas

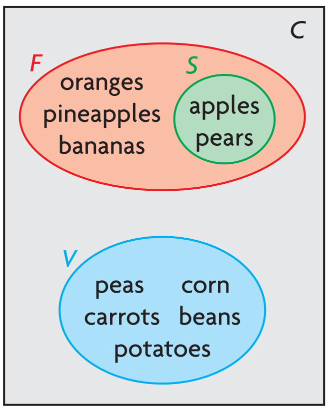
- You can represent a set of elements by:
  - listing the elements; for example, $A = \{1, 2, 3, 4, 5\}$
  - using words or a sentence; for example, $A = \{\text{all integers greater than 0 and less than 6}\}$
  - using set notation; for example, $A = \{x \mid 0 < x < 6, x \in \mathrm{I}\}$
- You can show how sets and their subsets are related using Venn diagrams. Venn diagrams do not usually show the relative sizes of the sets.
- You can often separate a universal set into subsets, in more than one correct way.

Need to Know



- Sets are equal if they contain exactly the same elements, even if the elements are listed in different orders.
- You may not be able to count all the elements in a very large or infinite set, such as the set of real numbers.
- The sum of the number of elements in a set and its complement is equal to the number of elements in the universal set:
$$n(A) + n(A') = n(U)$$
- When two sets A and B are disjoint,
$$n(A \text{ or } B) = n(A) + n(B)$$

CHECK Your Understanding



1. Imelda drew the Venn diagram to the left.

a) Imelda described the sets as follows:
- $C = \{\text{produce}\}$
- $F = \{\text{fruit}\}$
- $S = \{\text{fruit you can eat without peeling}\}$
- $V = \{\text{vegetables}\}$

Do you agree with her descriptions?

b) Describe another way Imelda could define the sets in her diagram.

c) Why does it make sense that $S \subset F$ and $S \subset C$?

d) List the disjoint sets, if there are any.

e) Is F' equal to V? Explain.

f) Determine $n(V)$ using $n(F)$ and $n(C)$.

g) List the elements in S'.

2. a) Draw a Venn diagram to represent these sets:
   - the universal set U = {natural numbers from 1 to 40 inclusive}
   - E = {multiples of 8}
   - F = {multiples of 4}
   - S = {multiples of 17}

   b) List the disjoint subsets, if there are any.

   c) Is each statement true or false? Explain.
      i) $E \subset F$
      ii) $F \subset E$
      iii) $E \subset E$
      iv) F' = {odd numbers from 1 to 40}
      v) In this example, the set of natural numbers from 41 to 50 is { }.

3. Nunavut (N) and the Northwest Territories (T) have the following fish species:
   - N = {walleye, northern pike, lake trout, Arctic char, Arctic grayling, lake whitefish}
   - T = {Arctic char, Arctic grayling, northern pike, lake trout, lake whitefish, inconnu, walleye}

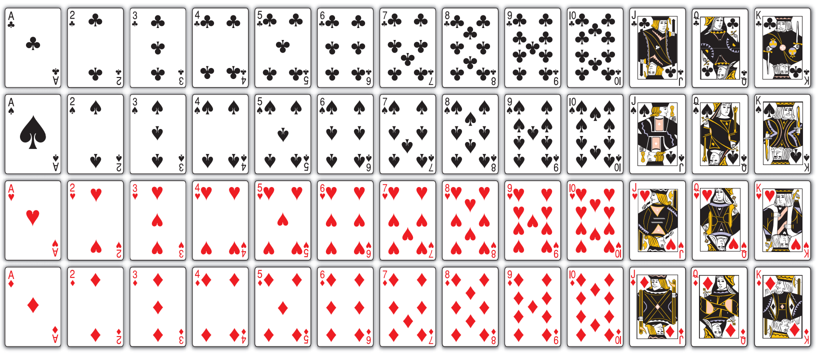
   a) Illustrate the sets of fish in these two territories using a Venn diagram.

   b) Explain what the following statement means: $N \subset T$, but $T \not\subset N$.

PRACTISING

4. For this question, the universal set, U, is a standard deck of 52 cards as shown.

   a) Represent the following sets and subsets using a Venn diagram:
   - B = {black cards}
   - R = {red cards}
   - S = {spades ♠}
   - H = {hearts ♥}
   - C = {clubs ♣}
   - D = {diamonds ♦}



   b) List all defined sets that are subsets of B.

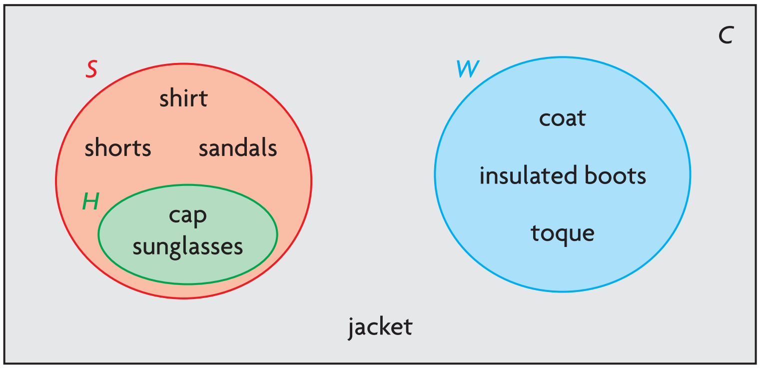
   c) List all defined sets that are subsets of R.

   d) Are sets S and C disjoint? Explain.

   e) Suppose you draw one card from the deck. Are the events drawing a heart and drawing a diamond mutually exclusive? Explain.

   f) Is the following statement correct?
      $n(S \text{ or } D) = n(S) + n(D)$
      Provide your reasoning. Determine the value of $n(S \text{ or } D)$.

5. Xavier drew this Venn diagram:



a) Describe what sets C, S, W, and H might represent.
b) Where would Xavier put running shoes?
c) Is S' equal to W? Explain.
d) List the disjoint sets, if there are any.
e) Categorize the items another way.

6. Consider the following information:
- the universal set $U = \{\text{natural numbers from 1 to 100 000}\}$
- $X \subset U$
- $n(X) = 12$

Determine $n(X')$, if possible. If it is not possible, explain why.

7. Consider the following information:
- the universal set $U = \{\text{all natural numbers from 1 to 10 000}\}$
- set X, which is a subset of U
- set Y, which is a subset of U
- $n(X) = 4500$

Determine $n(Y)$, if possible. If it is not possible, explain why.

8. Determine $n(U)$, the universal set, given $n(X) = 34$ and $n(X') = 42$.

9. Consider this universal set:
$A = \{$A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z$\}$

a) List the following subsets:
- $S = \{\text{letters drawn with straight lines only}\}$
- $C = \{\text{letters drawn with curves only}\}$

b) Is this statement true or false?
$C = S'$
Provide your reasoning.

10. Many forms of transportation can be used to travel around Alberta. These include walking, biking, driving, and skiing, as well as riding on buses, horses, airplanes, power boats, canoes, and hot-air balloons. Organize this information in a Venn diagram. Include other forms of transportation if you wish.

11. a) Organize the following sets of numbers in a Venn diagram:
   - $U = \{$integers from -10 to $10\}$
   - $A = \{$positive integers from 1 to 10 inclusive$\}$
   - $B = \{$negative integers from -10 to -1 inclusive$\}$

   b) List the disjoint subsets, if there are any.

   c) Is each statement true or false? Explain.
   - i) $A \subset B$
   - ii) $B \subset A$
   - iii) $A' = B$
   - iv) $n(A) = n(B)$
   - v) For set U, the set of integers from -20 to -15 is $\{ \}$.

12. Semiprime numbers are numbers that are the product of two prime numbers (1 is not considered to be a prime number). For example, 10 is a semiprime number because it is the product of 2 and 5.

   a) Use the set of natural numbers from 1 to 50, inclusive, as the universal set. Organize these numbers into the following sets:
   - $S = \{$semiprimes less than 50$\}$
   - $W = \{$other numbers$\}$

   b) Define one subset of S.

   c) Determine $n(W)$ without counting.

   d) Consider the set $A = \{$all semiprimes$\}$. Can you determine $n(A)$? Explain why or why not.

13. List items in your home that are related to entertainment or technology. Then organize these items into sets using a Venn diagram.

14. Cynthia claims that the $\subset$ sign for sets is similar to the $\leq$ sign for numbers. Explain whether you agree or disagree.

15. a) Indicate the multiples of 25 and 50, from -1000 to 1000, using set notation. List any subsets.

   b) Represent the sets and subsets in a Venn diagram.

16. Carol tosses a nickel, a dime, and a quarter. Each coin can turn up heads (H) or tails (T).

   a) List the elements of the universal set, U, for this situation.

   b) $E = \{$second coin turns up tails$\}$ List the elements of E.

   c) Determine $n(U)$ and $n(E)$.

   d) Is $E \subset U$?

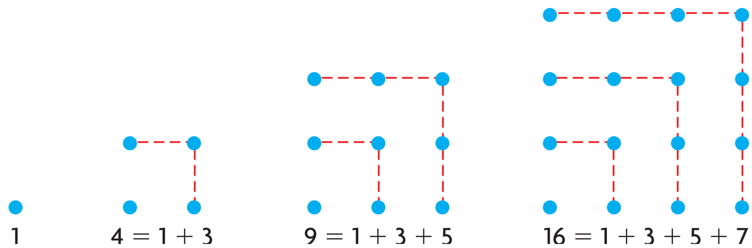
   e) Describe E' in words. Determine $n(E')$ using $n(U)$ and $n(E)$. Confirm your answer by listing the elements of E'.

   f) Are E and E' disjoint sets? Explain.

17. **a)** Organize the following sets in a Venn diagram:
- the universal set $R = \{\text{real numbers}\}$
- $N = \{\text{natural numbers}\}$
- $W = \{\text{whole numbers}\}$
- $I = \{\text{integers}\}$
- $Q = \{\text{rational numbers}\}$
- $\overline{Q} = \{\text{irrational numbers}\}$

b) Identify the complement of each set.
c) Identify any disjoint sets.
d) Are Q' and $\overline{Q}$ equal? Explain.
e) Of which sets is N a subset?
f) Joey drew a Venn diagram to show the sets in part a). In his diagram, the area of set Q was larger than the area of set $\overline{Q}$. Can you conclude that Q has more elements than $\overline{Q}$? Explain.

18. A square number can be represented as a square array.



Determine how many natural numbers from 1 to 300 inclusive are:
a) Even and square
b) Odd and square
c) Not square

Closing

19. Explain how to determine each of the following, and give an example.
a) Whether one set is a subset of another
b) Whether one set is a complement of another

Extending

20. Do you agree or disagree with the following explanation? Explain. Consider the set $U = \{\text{natural numbers from 20 to 30}\}$. One empty subset of U is the set of natural numbers from 1 to 19. Another empty subset is the natural numbers greater than 21. Therefore, U has two different empty subsets, not one.

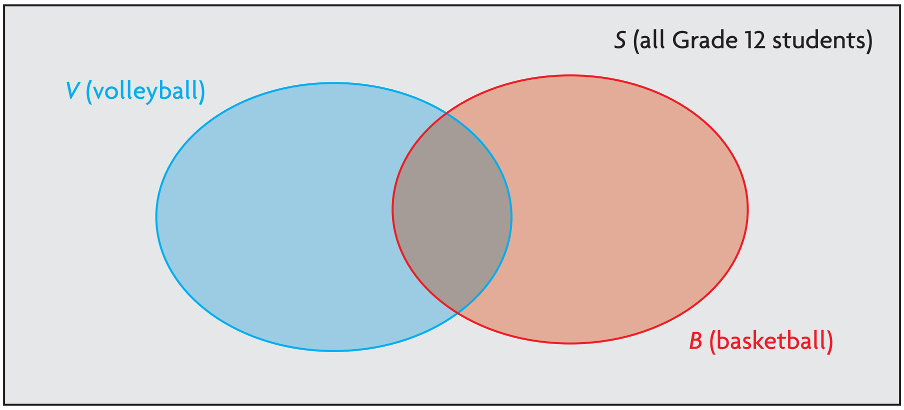
3.2 Exploring Relationships between Sets

GOAL

Explore what the different regions of a Venn diagram represent.

EXPLORE the Math

In an Alberta school, there are 65 Grade 12 students. Of these students, 23 play volleyball and 26 play basketball. There are 31 students who do not play either sport. The following Venn diagram represents the sets of students.



? How can you use this Venn diagram to determine the number of students who play volleyball only, basketball only, and both volleyball and basketball?

Reflecting

A. Consider the set of students who play volleyball and the set of students who play basketball. Are these two sets disjoint? Explain how you know.

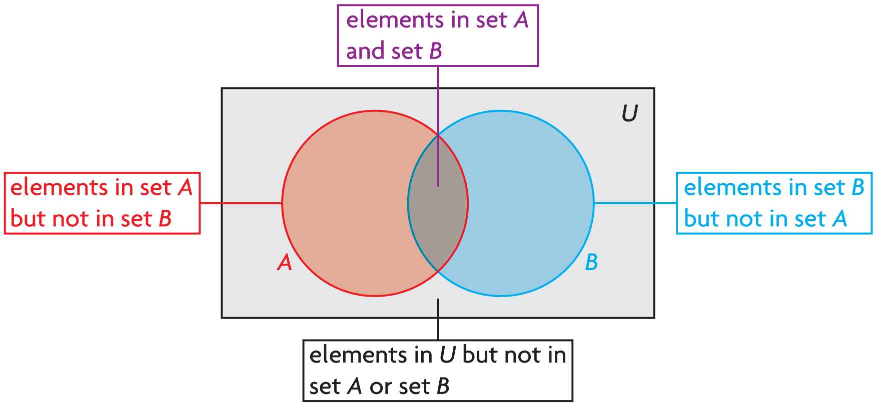
B. Describe how you solved the problem, including what each area of the Venn diagram represents.

C. Compare your solution to your classmates' solutions. Is there more than one way to solve the problem?

In Summary

Key Ideas

- Sets that are not disjoint share common elements.
- Each area of a Venn diagram represents something different.
- When two non-disjoint sets are represented in a Venn diagram, you can count the elements in both sets by counting the elements in each region of the diagram just once.



Need to Know

- Each element in a universal set appears only once in a Venn diagram.
- If an element occurs in more than one set, it is placed in the area of the Venn diagram where the sets overlap.

FURTHER Your Understanding

1. Consider the following sets:

- $U = \{2, 3, 4, 6, 8, 9, 10, 12, 14, 15\}$
- $A = \{3, 6, 9, 12, 15\}$
- $B = \{2, 4, 6, 8, 10, 12, 14\}$

a) Illustrate these sets using a Venn diagram.

b) Determine the number of elements

i) in set A.

ii) in set A but not in set B.

iii) in set B.

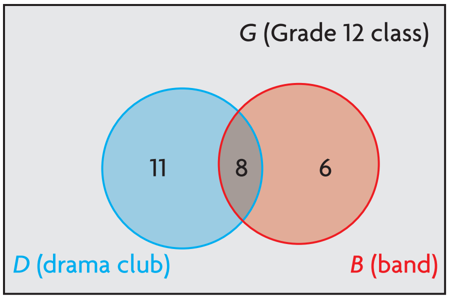
iv) in set B but not in set A.

v) in set A **and** set B.

vi) in set A **or** set B.

vii) in A'.

2. There are 38 students in a Grade 12 class. The number of students in the drama club and the band are illustrated in the Venn diagram. Use the diagram to answer the following questions.



   a) How many students are in both the drama club and the band?
   b) How many students are in the drama club but not in band? How many are in the band but not in the drama club?
   c) How many students are in the drama club? How many are in the band?
   d) How many students are in at least one of the drama club or the band?
   e) How many students are in neither the drama club nor the band?

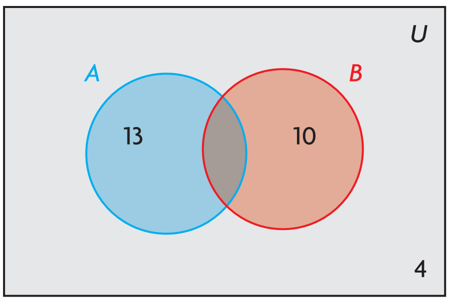
3. Anna surveyed 45 students about their favourite sports. She recorded her results.

Favourite Sports	Number of Students
hockey	20
soccer	14
neither hockey nor soccer	16

   a) Determine how many students like hockey and soccer.
   b) Determine how many students like only hockey or only soccer.
   c) Draw and label a Venn diagram to show the data.

4. There are 55 guests at a ski resort in British Columbia. Of these guests, 25 plan to go skiing and 32 plan to go snowboarding. There are 9 guests who do not plan to ski or snowboard.
   a) Determine how many guests plan to ski and snowboard.
   b) Determine how many guests plan to only ski.
   c) Determine how many guests plan to only snowboard.

5. Ryan drew the following Venn diagram incorrectly. There are 25 items in the universal set, U, and 4 items that are not in set A or set B.



   a) Determine $n(A \text{ and } B)$, $n(A \text{ only})$, and $n(B \text{ only})$.
   b) Redraw Ryan's Venn diagram with the data you determined in part a).

3.3 Intersection and Union of Two Sets

EXPLORE...

Given: $n(A) = x$, $n(B) = y$, $n(A \text{ and } B)' = \{\ \}$, and $n(U) = z$, where U = the universal set, and sets A and B are subsets of U.

- How can you determine whether sets A and B are disjoint or overlap?

Communication | *Tip*

The 24-hour clock goes from 0 to 24:00 each day; for example, 2:00 a.m. is 02:00 and 5:00 p.m. is 17:00. The times 00:00 and 24:00 both represent midnight.

GOAL

Understand and represent the intersection and union of two sets.

INVESTIGATE the Math

Jacquie is a zookeeper. She is responsible for feeding any baby animal that cannot be fed by its mother. She needs to feed a baby raccoon every 2 h and a baby lemur every 3 h. She uses the 24-hour clock to plan their feeding times. She starts to feed the raccoon at 02:00 and the lemur at 03:00.

? At what times will Jacquie need help to feed the raccoon and the lemur?

A. Let set T represent all the hourly times from 01:00 to 24:00.
Let set R represent times for every 2 h, up to 24 h.
Let set L represent times for every 3 h, up to 24 h.
Describe the universal set T and the subsets R and L in set notation.

B. List the elements in each subset.

C. Represent the sets T, R, and L using a Venn diagram. Place the feeding times in the correct regions. Explain why you placed the sets and elements where you did.

D. List the elements in the **intersection** of sets R and L, $R \cap L$.

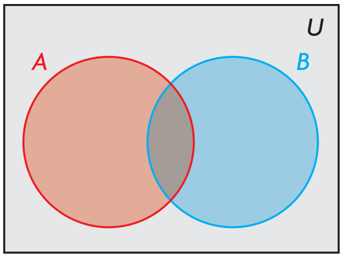
E. List the elements in set $R \setminus L$ and in set $L \setminus R$ using set notation.

intersection

The set of elements that are common to two or more sets. In set notation, $A \cap B$ denotes the intersection of sets A and B; for example, if $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $A \cap B = \{3\}$.

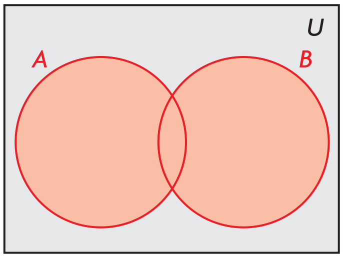
Communication | *Notation*

In set notation, $A \cap B$ is read as "intersection of A and B." It denotes the elements that are common to A and B. The intersection is the region where the two sets overlap in the Venn diagram below.



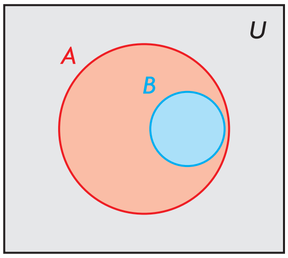
$A \cap B$

$A \cup B$ is read as "union of A and B." It denotes all elements that belong to at least one of A or B. The union is the red region in the Venn diagram below.

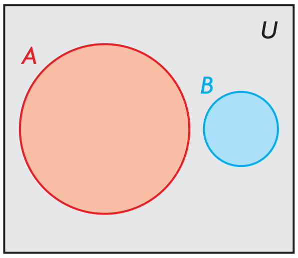


$A \cup B$

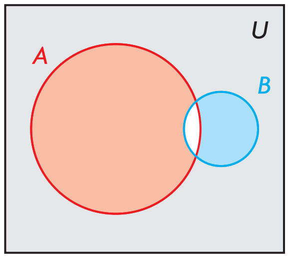
$A \setminus B$ is read as "A minus B." It denotes the set of elements that are in set A but not in set B. It is the red region in each Venn diagram below.



$A \setminus B$ when $B \subset A$



$A \setminus B$ when they are disjoint



$A \setminus B$ when they intersect

F. The set of times that belong to **either** a feeding schedule of every 2 h **or** a feeding schedule of every 3 h forms the **union** of sets R and L, or $R \cup L$. List the elements in $R \cup L$ using set notation.

union

The set of all the elements in two or more sets; in set notation, $A \cup B$ denotes the union of sets A and B; for example, if $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $A \cup B = \{1, 2, 3, 4, 5\}$.

G. List the elements in $(R \cup L)'$, the complement of the union of R and L. Include these elements in your Venn diagram.

H. Complete each statement with "and" or "or."
- The set $R \cap L$ consists of the elements in set R _____ set L.
- The set $R \cup L$ consists of the elements in set R _____ set L.

I. How does your Venn diagram show the times when Jacquie will need help to feed the two baby animals?

Reflecting

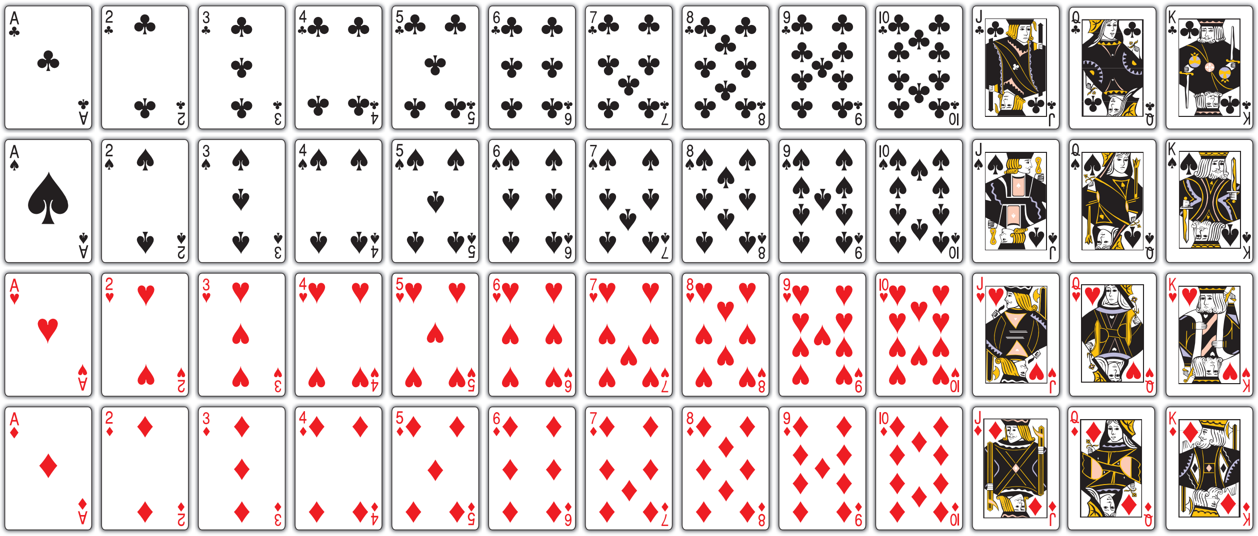
J. Explain whether you agree or disagree with the following statement: The union of any two sets is like the addition of two numbers, so $n(R \cup L) = n(R) + n(L)$. If you disagree with this statement, write the correct formula for $n(R \cup L)$.

K. Write a formula for $n(L \setminus R)$, the number of elements that are in set L but not in set R. Explain your formula. Will your formula also work for disjoint sets?

APPLY the Math

EXAMPLE 1 Determining the union and intersection of disjoint sets

If you draw a card at random from a standard deck of cards, you will draw a card from one of four suits: clubs (C), spades (S), hearts (H), or diamonds (D).



a) Describe sets C, S, H, and D, and the universal set U for this situation.
b) Determine $n(C)$, $n(S)$, $n(H)$, $n(D)$, and $n(U)$.
c) Describe the union of S and H. Determine $n(S \cup H)$.
d) Describe the intersection of S and H. Determine $n(S \cap H)$.
e) Determine whether the events that are described by sets S and H are mutually exclusive, and whether sets S and H are disjoint.
f) Describe the complement of $S \cup H$.

Petra's Solution

a) U = {drawing a card from a deck of 52 cards}
S = {drawing a spade ♠}
H = {drawing a heart ♥}
C = {drawing a club ♣}
D = {drawing a diamond ♦}

I described the sets.

b) $n(U) = 52$
$n(S) = 13$
$n(H) = 13$
$n(C) = 13$
$n(D) = 13$

I used the deck of cards to count the number of elements in each set.

c) $S \cup H = \{$the set of 13 spades and of 13 hearts$\}$
$n(S \cup H) = 26$

The union of S and H consists of 26 cards, either spades or hearts.

d) $S \cap H = \{\ \}$
The events described by S and H are mutually exclusive.
$n(S \cap H) = 0$

A card cannot be a spade and a heart, so sets S and H have no common elements. Their intersection is the empty set.

The events that are described by sets S and H (drawing a heart and drawing a spade) must be mutually exclusive.

e) Since the events described by sets S and H are mutually exclusive, these sets are disjoint.

There is no intersection of sets S and H, so these sets are disjoint.

f) $(S \cup H)' = \{$the set of cards that are not hearts or spades, or the set of clubs and diamonds$\}$
$(S \cup H)' = (C \cup D)$

The complement of $S \cup H$ contains the cards in the deck that are not in S or H. These are clubs and diamonds.

Since clubs and diamonds are also disjoint, I can write $(S \cup H)'$ as the union of C and D.

Your Turn

Petra thinks that $n(S) + n(H) = n(S \cup H)$. Is she correct? Explain.

EXAMPLE 2 Determining the number of elements in a set using a formula

The athletics department at a large high school offers 16 different sports:

badminton	hockey	tennis
basketball	lacrosse	ultimate
cross-country running	rugby	volleyball
curling	cross-country skiing	wrestling
football	soccer	
golf	softball	

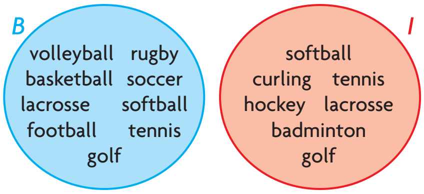
Determine the number of sports that require the following types of equipment:

a) a ball and an implement, such as a stick, a club, or a racquet
b) only a ball
c) an implement but not a ball
d) either a ball or an implement
e) neither a ball nor an implement

Terry's Solution

a) $U = \{$sports offered by the athletics department$\}$
$B = \{$sports that use a ball$\}$
$I = \{$sports that use an implement$\}$

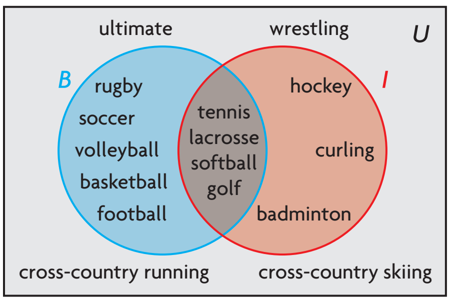
I defined the universal set, U, and the subsets I would use.



$n(B) = 9$ $n(I) = 7$

I could have just counted the sports, but I wanted to use an organized method to make sure that I accounted for each sport.

I started to draw a Venn diagram. I drew one circle, B, for sports that use a ball. I drew another circle, I, for sports that use an implement.



Number of elements in the intersection:

$n(B \cap I) = 4$

Therefore, 4 sports use a ball and an implement.

I realized my first diagram could be improved since golf, lacrosse, softball, and tennis were in both circles, so I overlapped the circles and put these sports in the overlap.

Sports that use neither a ball nor an implement are in the box outside the circles. U is the universal set of all sports offered at this school.

I counted. There are 4 sports in the intersection.

b) Number of elements in B minus I:

$n(B \setminus I) = n(B) - n(B \cap I)$
$n(B \setminus I) = 9 - 4$
$n(B \setminus I) = 5$

Therefore, 5 sports use only a ball.

Sports that use only a ball are in the blue circle, but not in the intersection.

c) Number of elements in I minus B:

$n(I \setminus B) = n(I) - n(B \cap I)$
$n(I \setminus B) = 7 - 4$
$n(I \setminus B) = 3$

Therefore, 3 sports use an implement but not a ball.

Sports that use an implement but not a ball are in the red circle, but not in the intersection.

d) Number of elements in union of I and B

$n(I \cup B) = n(B) + n(I) - n(B \cap I)$
$n(I \cup B) = 9 + 7 - 4$
$n(I \cup B) = 12$

Therefore, 12 sports use either a ball or an implement.

Sports that use either a ball or an implement are in the union of the blue and red circles.

I used the **Principle of Inclusion and Exclusion** to determine the number of elements in the union of I and B.

e) Number of elements in the union of the complement of B and I:

$n((B \cup I)') = n(U) - n(B \cup I)$
$n((B \cup I)') = 16 - 12$
$n((B \cup I)') = 4$

Therefore, 4 sports use neither a ball nor an implement.

Sports that use neither a ball nor an implement lie outside the circles. The number of these sports is the complement of $n(I \cup B)$.

I confirmed all my results by counting on my Venn diagram.

Principle of Inclusion and Exclusion

The number of elements in the union of two sets is equal to the sum of the number of elements in each set, less the number of elements in both sets; using set notation, this is written as $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

Your Turn

The high school in Example 2 now offers water polo. Organize all the sports in a Venn diagram according to the special headgear and footwear they require. What questions can you answer using your Venn diagram?

EXAMPLE 3 Determining the number of elements in a set by reasoning

Jamaal surveyed 34 people at his gym. He learned that 16 people do weight training three times a week, 21 people do cardio training three times a week, and 6 people train fewer than three times a week.
How can Jamaal interpret his results?

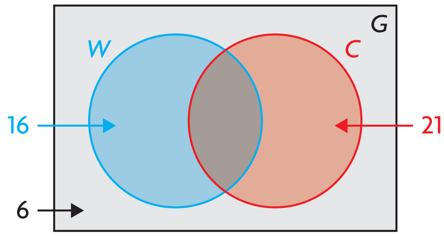
Jamaal's Solution

Number surveyed $= 34$
Sum of survey data $= 16 + 21 + 6$ or 43

Since there are more replies than people surveyed, I knew that some people do both types of exercise three times a week.

$G = \{$all the people surveyed at the gym$\}$
$W = \{$people who do weight training$\}$
$C = \{$people who do cardio training$\}$

I defined the universal set G and subsets W and C.

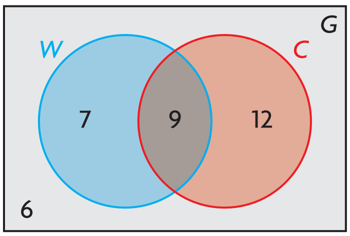


I drew a Venn diagram. I drew two circles for people who exercise three times a week.

I knew that the 6 people who train fewer than three times a week are in neither circle.

Of the 34 people I surveyed, 6 train less than three times a week. So, 34 – 6 or 28 people exercise three times a week. Of these people, 16 do weight training and 21 do cardio training.

I represented the number of people in each region in my Venn diagram.



I learned that $16 + 21$ or 37 people do either one or both types of training. So, $37 - 28$ or 9 people must do both types of training. I wrote 9 where the circles intersect.

Since 16 people are in set W, $16 - 9$ or 7 people must be only in set W.

Similarly, $21 - 9$ or 12 people are only in set C.

9 people do cardio training three times a week and do weight training three times a week.
7 people do only weight training three times a week.
12 people do only cardio training three times a week.
6 people train fewer than three times a week.

I summarized my results.

Your Turn

Jamaal surveyed 50 other gym members. Of these members, 9 train fewer than three times a week, 11 do cardio training three times a week, and 16 do both cardio and weight training three times a week. Determine how many of these members do weight training three times a week.

EXAMPLE 4 Correcting errors in Venn diagrams

Morgan surveyed the 30 students in her mathematics class about their eating habits.

- 18 of these students eat breakfast.
- 5 of the 18 students also eat a healthy lunch.
- 3 students do not eat breakfast and do not eat a healthy lunch.

How many students eat a healthy lunch?

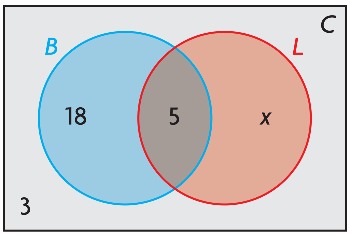
Tyler solved this problem, as shown below, but made an error. What error did Tyler make? Determine the correct solution.

Tyler's Solution

Let C represent the universal set, the students in Morgan's mathematics class. Let B represent those who eat breakfast, and let L represent those who eat a healthy lunch.

There are 30 students in total.

I drew a Venn diagram showing the number of elements in each region.



There are 18 students in set B. I put the 5 students who are in sets B and L in the overlap.

There are 3 students who do not belong in either circle. This means there are $30 - 3$ or 27 people in the coloured regions.

$$18 + 5 + x = 27$$
$$x = 4$$

I determined the total number of elements in set L.

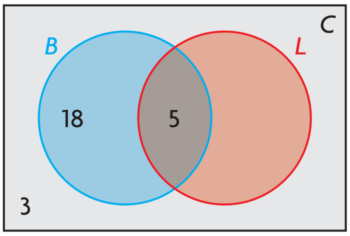
$$n(L) = 5 + 4$$
$$n(L) = 9$$

Therefore, 9 students eat a healthy lunch.

Vanessa's Solution

Tyler erred in interpreting the data and placing it on the Venn diagram.

He assumed that 18 students ate breakfast but not a healthy lunch.



$n(C) = 30$
$n(B \cup L) = n(C) - n(B \cup L)'$
$n(B \cup L) = 30 - 3$
$n(B \cup L) = 27$

I knew the number of students surveyed.

The set of students who do not eat breakfast or a healthy lunch is $(B \cup L)'$.

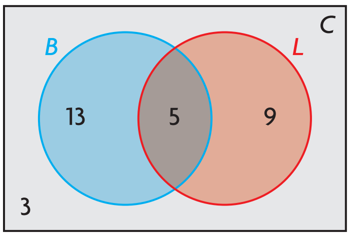
I determined $n(B \cup L)$.

$n(B) = 18$
$n(B \cap L) = 5$
$n(B \setminus L) = n(B) - n(B \cap L)$
$n(B \setminus L) = 18 - 5$
$n(B \setminus L) = 13$

I knew that 18 students eat breakfast and 5 of them also eat a healthy lunch. I determined the number of students who eat breakfast but not a healthy lunch.

$n(B \cup L) = n(B \setminus L) + n(L \setminus B) + n(B \cap L)$
$27 = 13 + n(L \setminus B) + 5$
$9 = n(L \setminus B)$

I determined the number of students who eat a healthy lunch but do not eat breakfast by adding the number of elements in the three regions of the Venn diagram that involve sets B and L.



I redrew Tyler's Venn using the correct numbers of elements.

$n(L) = n(L \setminus B) + n(B \cap L)$
$n(L) = 9 + 5$
$n(L) = 14$

I determined the total in set L.

Therefore, 14 students eat a healthy lunch.

Your Turn

Susan surveyed the 34 students in her science class.

- 14 students eat breakfast.
- 16 students eat a healthy lunch.
- 4 of the 30 students above eat breakfast and a healthy lunch.

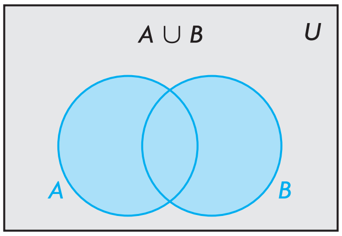
Since $14 + 16 + 4 = 34$, Susan concluded that everyone eats either breakfast or a healthy lunch, or both.

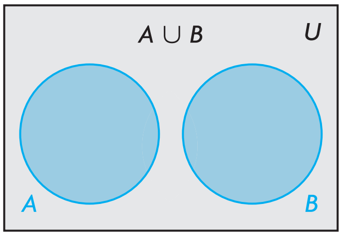
What error did Susan make? How many students do not eat either meal?

In Summary

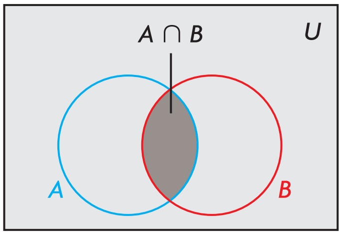
Key Ideas

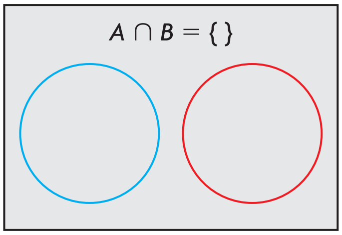
- The **union** of two or more sets, for example, $\boldsymbol{A \cup B}$, consists of all the elements that are in at least one of the sets. It is represented by the entire region of these sets on a Venn diagram. It is indicated by the word "**or**."





- The **intersection** of two or more sets, for example, $\boldsymbol{A \cap B}$, consists of all the elements that are common to these sets. It is represented by the region of overlap on a Venn diagram. It is indicated by the word "**and**."



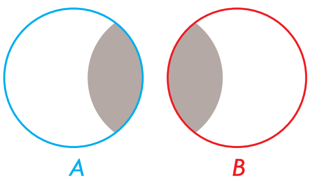


Need to Know

- If two sets, A and B, contain common elements, the number of elements in A or B, $n(A \cup B)$, is:

  $$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

  This is called the Principle of Inclusion and Exclusion. To calculate $n(A \cup B)$, subtract the elements in the intersection so they are not counted twice, once in $n(A)$ and once in $n(B)$.



- If two sets, A and B, are disjoint, they contain no common elements:

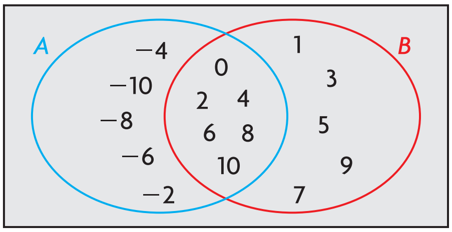
  $$n(A \cap B) = 0 \text{ and}$$
  $$n(A \cup B) = n(A) + n(B)$$

- Elements that are in set A but not in set B are expressed as $A \setminus B$. The number of elements in A or B, $n(A \cup B)$, can also be determined as follows:

  $$n(A \cup B) = n(A \setminus B) + n(B \setminus A) + n(A \cap B)$$

CHECK Your Understanding

1. Consider the following Venn diagram:
   a) Determine $A \cup B$.
   b) Determine $n(A \cup B)$.
   c) Determine $A \cap B$.
   d) Determine $n(A \cap B)$.



The Canadian lynx prefers dense forest.

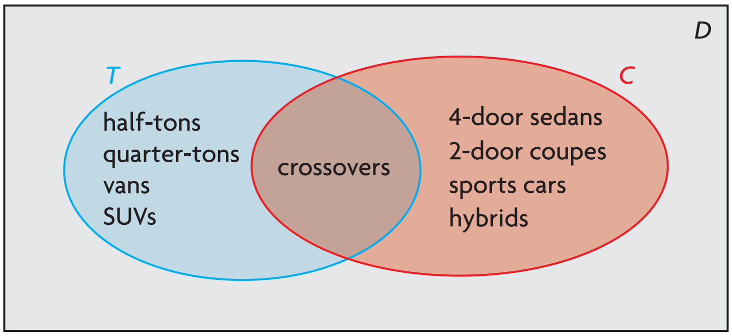
2. Animals that are native to the tundra, the ecosystems of Canada's Far North, include the Arctic fox, caribou, ermine, grizzly bear, muskox, and polar bear. The taiga, ecosystems south of the tundra, consists primarily of coniferous forest. Animals that are native to the taiga include the bald eagle, Canadian lynx, grey wolf, grizzly bear, long-eared owl, and wolverine.
   a) Determine the union and intersection of these two sets of animals.
   b) Draw a Venn diagram of these two sets.

3. Consider the following two sets:
   - $A = \{-10, -8, -6, -4, -2, 0, 2, 4, 6, 8, 10\}$
   - $C = \{2, 4, 6, 8, 10, 12, 14, 16\}$

   a) Determine $A \cup C$, $n(A \cup C)$, $A \cap C$, and $n(A \cap C)$.
   b) Draw a Venn diagram to show these two sets.

PRACTISING

4. The following Venn diagram shows the types of vehicles at a car dealership:



The takin is the national animal of Bhutan.

   a) Determine $T \cup C$.
   b) Determine $n(T \cup C)$.
   c) Determine $T \cap C$.

5. Animals that are native to Africa include the lion, camel, giraffe, hippopotamus, and elephant. Animals that are native to Asia include the elephant, tiger, takin, and camel.
   a) Draw a Venn diagram to show these two sets of animals.
   b) Determine the union and intersection of these two sets.

6. Consider the following two sets:
   - $A = \{j \mid j = 2x, -3 \leq x \leq 6, x \in \mathrm{I}\}$
   - $B = \{k \mid k = 3x, -4 \leq x \leq 5, x \in \mathrm{I}\}$

   a) Draw a Venn diagram to show these two sets.
   b) Determine $A \cup B$, $n(A \cup B)$, $A \cap B$, and $n(A \cap B)$.

Sherlock Holmes and Hercule Poirot are fictional characters.

7. Rosie asked 25 people at a mystery convention if they liked Sherlock Holmes or Hercule Poirot.
   - 4 people did not like either detective.
   - 16 people liked Sherlock Holmes.
   - 11 people liked Hercule Poirot.

   Determine how many people liked both detectives, how many liked only Sherlock Holmes, and how many liked only Hercule Poirot.

8. Tashi asked 80 people if they liked vanilla or chocolate ice cream.
   - 9 people did not like either flavour.
   - 11 people liked both vanilla and chocolate.
   - 20 people liked only vanilla.

   Determine how many people liked only chocolate.

9. John asked 26 people at a gym if they liked to ski or swim.
   - 5 people did not like to do either sport.
   - 19 people liked to ski.
   - 14 people liked to swim.

   Determine how many people liked to ski and swim.

10. Tiffany volunteers in an elementary classroom. She is helping students understand multiples of 2 and 3 in mathematics. The students are working with the numbers 1 to 30. How can Tiffany use a Venn diagram to show the students how the multiples relate to one another?

11. Mark surveyed 100 people at a local doughnut shop.
    - 65 people ordered coffee.
    - 45 people ordered a doughnut.
    - 10 people ordered something else.

    Mark wants to determine how many people ordered coffee and a doughnut.

    a) Model this situation with sets. Identify the universal set, and explain what subsets you will use.
    b) Draw a Venn diagram to model this situation. Explain what each part of your diagram represents.
    c) Determine how many people ordered coffee and a doughnut.

12. At a retirement home, 100 seniors were interviewed.
- 16 seniors like to watch television and listen to the radio.
- 67 seniors like to watch television.

Determine how many seniors prefer to listen to the radio only.

13. In Edmonton, Anita asked 56 people if they had been to the Calgary Stampede or the Pacific National Exhibition (PNE).
- 14 people had not been to either.
- 30 people had been to the Calgary Stampede.
- 22 people had been to the PNE.

Determine how many people had been to both events.

14. Armour is a real estate agent. He asked 54 clients where they live now.
- 31 people own their home.
- 30 people live in a condominium.
- 9 people rent their house.

Determine how many people own the condominium in which they live.

15. Jamal asked 32 people what type of television shows they like.
- 13 people like reality shows but not contest shows.
- 9 people like contest shows but not reality shows.
- 4 people like neither type of show.

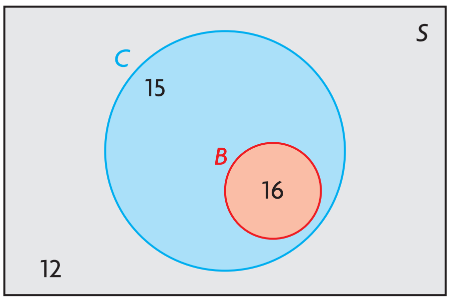
Determine how many people like both types of shows.

16. Beyondé solved the following problem:

A total of 48 students were asked how they got to school.
- 31 students drive a car.
- 16 students take a bus.
- 12 students do not drive a car or take a bus.
- Some students drive a car or take a bus.

Determine how many students do not take a bus to school.

Beyondé's Solution



15 students drive a car but do not take a bus, 12 students do neither. So, 27 students do not take a bus.

The total of the three numbers is 59. So, I knew that the region for students who take a bus overlaps the region for students who drive a car.

I drew a Venn diagram with 31 students in the car region and 16 students in the bus region.

Is Beyondé correct? Justify your answer.

17. Given:

$n(A) + n(B) = n(A \cup B)$ and

$n(A) + n(C) > n(A \cup C)$

a) Which sets do you know are disjoint?

b) Which sets do you know intersect?

c) Are there any sets that could either be disjoint or intersect? If so, which sets? Explain.

The Arctic Winter Games give youth who live in the North the opportunity to meet each other. Historically, participants come from Yukon, Northwest Territories, Nunavut, Northern Alberta, Northern Québec, Alaska, Greenland, Russia, and the Sami people of northern Europe.

Closing

18. Which is more like the addition of two numbers: the union of two sets or the intersection of two sets? Explain.

Extending

19. The Arctic Winter Games include alpine skiing, cross-country skiing, free-style skiing, badminton, basketball, snowshoe biathlon, ski biathlon, curling, dog mushing, figure skating, gymnastics, hockey, indoor soccer, snowboarding, snowshoeing, speed skating, table tennis, volleyball, and wrestling. There are also two categories of sports that are unique to the Arctic, called Arctic Sports and Dene Games.

a) Determine a way to sort the games into sets and subsets.

b) List each set and subset.

c) Draw a Venn diagram to illustrate the sets.

d) Compare your results with your classmates' results. Is there more than one way to sort the games?

History | Connection

Unexpected Infinities

Things do not always turn out the way you might expect. For example, most people would expect two infinite sets to be the same "size." But Georg Cantor, who developed set theory, showed that the set of natural numbers and the set of real numbers contain a *different* number of elements.

A. Set theory gives rise to other situations that are paradoxical or counterintuitive (not what you would expect). For example, explore the "barber paradox" on the Internet.

B. Research other paradoxes and counterintuitive ideas associated with set theory.

Georg Cantor (1845–1918) was born in Russia and spent his career teaching and researching at German universities.

3 Mid-Chapter Review

FREQUENTLY ASKED Questions

Study *Aid*
- See Lesson 3.1, Example 1.
- Try Mid-Chapter Review Questions 1 to 3.

Q: How can you represent the elements in a set?

A: If the set is finite, you can do the following:
- Describe the set. For example, $E = \{\text{even numbers between 2 and 10}\}$.
- List the elements in any order. For example, the set of days that begin with T can be represented as $T = \{\text{Thursday, Tuesday}\}$.
- Use set notation. For example, the set of positive integers less than 1000 can be represented as follows:

$$P = \{x \mid 0 < x < 1000, x \in I\}$$

If a set is infinite, you can either describe it or use set notation. For example:

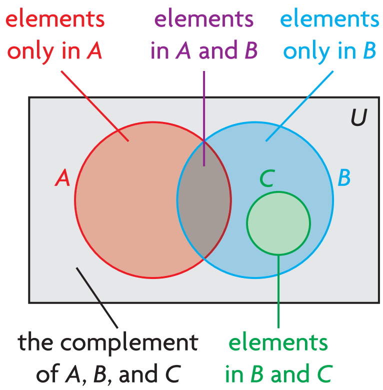
$$L = \{\text{the set of rational numbers greater than or equal to 25}\}, \text{ or}$$
$$L = \{k \mid k \geq 25, k \in Q\}$$

You can choose any letter to name a set.

Study *Aid*
- See Lesson 3.1, Examples 1, 3, and 4, and Lesson 3.2.
- Try Mid-Chapter Review Questions 2 to 5.

Q: How does a Venn diagram show how sets are related?

A: For example, in the Venn diagram below, sets A and B overlap because they share common elements. Set C is inside set B because it is a subset of set B. Sets A and C do not overlap because they do not share any elements; that is, they are disjoint. The complement of a set, or sets, consists of all the elements of the universal set that are not in the original set. A Venn diagram shows the relationships between sets but not their relative sizes.



Q: What is meant by the intersection and union of two sets?

A: The intersection of two sets, A and B, consists of the elements that are common to both sets.
It is represented as $A \cap B$ and is read as "A and B."
If there are no common elements, the intersection is the empty set.

The union of two sets is the elements in the first set only, the second set only, and the intersection of both sets. It is represented as $A \cup B$ and read as "A or B."

For example, consider the following two sets of playing cards:

$H = \{A♥, 3♥, 5♥, 7♥, 9♥, J♥, Q♥, K♥\}$

$F = \{J♠, Q♠, K♠, J♦, Q♦, K♦, J♣, Q♣, K♣, J♥, Q♥, K♥\}$

The intersection:
$H \cap F = \{J♥, Q♥, K♥\}$

The union:
$H \cup F = \{A♥, 3♥, 5♥, 7♥, 9♥, J♥, Q♥, K♥, J♠, Q♠, K♠, J♦, Q♦, K♦, J♣, Q♣, K♣\}$

Study Aid

- See Lesson 3.3, Examples 1 and 2.
- Try Mid-Chapter Review Questions 6 and 7.

Q: How can you determine the number of elements in the union of two sets?

A: Use the Principle of Inclusion and Exclusion:

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

If the sets are disjoint, the intersection is the empty set. For example, consider the sets in the previous example:
$n(H) = 8$, $n(F) = 12$, and $n(H \cap F) = 3$
$n(H \cup F) = 8 + 12 - 3$ or 17

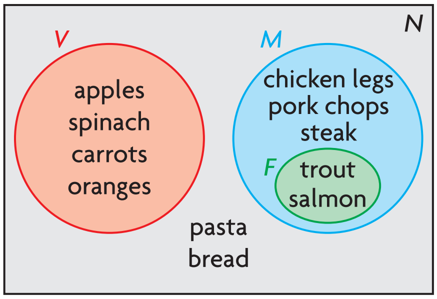
Study Aid

- See Lesson 3.3, Examples 3 and 4.
- Try Mid-Chapter Review Questions 6 and 7.

PRACTISING

Lesson 3.1

1. Lucy drew the following Venn diagram:



a) Using set notation and sets V, M, F, and N, list the subsets.
b) How might Lucy have chosen what to put in each set?
c) Is M' equal to V? Explain.
d) List the disjoint sets, if there are any.

2. a) Draw a Venn diagram to show:
- the universal set $U = \{\text{natural numbers from 1 to 40}\}$
- $E = \{\text{multiples of 3}\}$
- $F = \{\text{multiples of 15}\}$
- $S = \{\text{multiples of 9}\}$

b) List the disjoint subsets, if there are any.
c) Is each statement true or false? Explain.
i) $E \subset F$
ii) $S \subset E$
iii) $S \subset F$
iv) $F' = \{\text{all numbers from 1 to 40 except 15 and 30}\}$
v) $S \subset S$

3. Make a list of different sports equipment. Organize the equipment into sets using a Venn diagram.

Lesson 3.2

4. Jordan asked 40 students at his school cafeteria what they bought for lunch. He recorded his results in the table below.

Purchase	Number of Students
beverage	34
soup	18
no beverage or soup	5

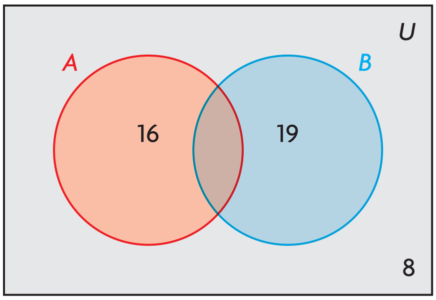
a) How many students bought a beverage and soup?
b) How many students bought only a beverage or only soup?
c) Draw a Venn diagram to show the data.

5. A total of 20 students are on a field trip. Of these students, 13 are wearing sunglasses and 6 are wearing a hat. Only 5 students are not wearing sunglasses or a hat.
a) How many students are wearing both sunglasses and a hat?
b) How many students are wearing sunglasses but not a hat?
c) How many students are wearing a hat but not sunglasses?

Lesson 3.3

6. Tanya was given the following sets and asked to represent them using a Venn diagram:
- U is the universal set.
- A and B are subsets of U.
- $n(U) = 40$, $n(A) = 16$, and $n(B) = 19$
- $n((A \cup B)') = 8$

She drew the following Venn diagram:



a) Tanya made an error in her Venn diagram. What was her error? Explain.
b) Redraw Tanya's Venn diagram correctly.

7. Paul asked 20 students whether they have a dog or a cat.
- 4 students do not have a dog or a cat.
- 8 students have a dog.
- 8 students have a cat.

How many students have both a dog and a cat?

3.4 Applications of Set Theory

GOAL

Use sets to model and solve problems.

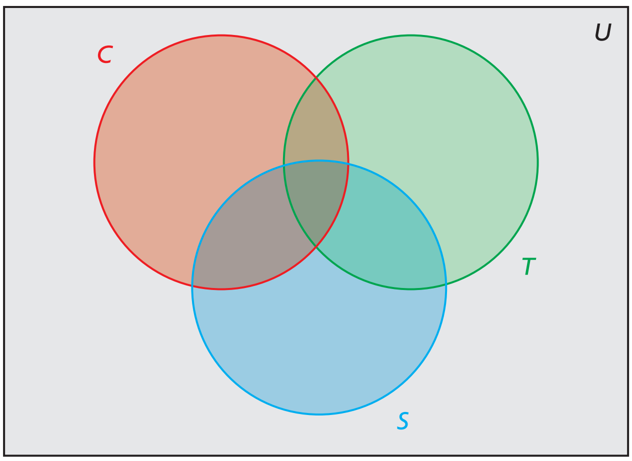
INVESTIGATE the Math

Rachel surveyed Grade 12 students about how they communicated with friends over the previous week.

- 66% called on a cellphone.
- 76% texted.
- 34% used a social networking site.
- 56% called on a cellphone and texted.
- 18% called on a cellphone and used a social networking site.
- 19% texted and used a social networking site.
- 12% used all three forms of communication.

? What percent of students used at least one of these three forms of communication?

A. The Venn diagram below represents the following sets:
- $C =$ {students who called on a cellphone}
- $T =$ {students who texted}
- $S =$ {students who used a social networking site}



i) What does the universal set U represent in this situation?

ii) Copy this Venn diagram. Record the percent of students who used all three forms of communication on your diagram.

EXPLORE...

Sarah conducted a survey of teen gaming preferences. Here are her results:
- 20 teens play online games.
- 20 play on a game console.
- 20 play games on their cellphone.

She surveyed only 31 teens. How can this be?

B. Determine the percent of students who texted and used a social networking site, but did not call on a cellphone. Update your diagram.

C. Determine the percent of students who called on a cellphone and used a social networking site, but did not text. Determine the percent of students who called on a cellphone and texted, but did not use a social networking site. Update your diagram.

D. Determine the percent of students who only called on a cellphone, only texted, or only used a social networking site. Update your diagram.

E. Determine the percent of students who used at least one of these three forms of communication. Explain your answer.

Reflecting

F. Serge claims that the Principle of Inclusion and Exclusion can be used to develop a formula for $n(S \cup T \cup C)$ as follows:

$$n(S \cup T \cup C) = n(S) + n(T) + n(C) - n(S \cap T) - n(S \cap C) - n(T \cap C) + n(S \cap T \cap C)$$

Does this formula give the same answer you got in part E? Explain.

G. Determine the percent of students who called on a cellphone or texted, but did not use a social networking site. Express your result in set notation.

H. How would your Venn diagram change if 16% of the students had used all three forms of communication?

APPLY the Math

EXAMPLE 1 Solving a puzzle using the Principle of Exclusion and Inclusion

Use the following clues to answer the questions below:

- 28 children have a dog, a cat, or a bird.
- 13 children have a dog.
- 13 children have a cat.
- 13 children have a bird.
- 4 children have only a dog and a cat.
- 3 children have only a dog and a bird.
- 2 children have only a cat and a bird.
- No child has two of each type of pet.

a) How many children have a cat, a dog, and a bird?
b) How many children have only one pet?

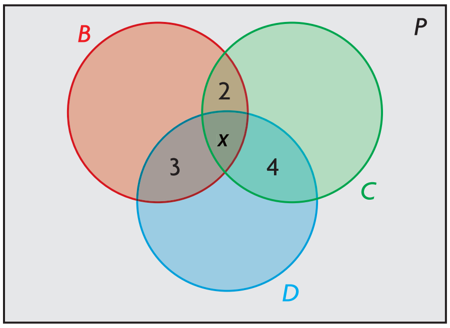
Hailey's Solution: Using the Principle of Inclusion and Exclusion

a) $P = \{\text{children with pets}\}$ $C = \{\text{children with a cat}\}$
$B = \{\text{children with a bird}\}$ $D = \{\text{children with a dog}\}$

I defined the sets in this situation.

Let x represent the number of children with a bird, a cat, and a dog.

I defined the variable x.



I drew a Venn diagram to show how the numbers of elements in the four sets were related.

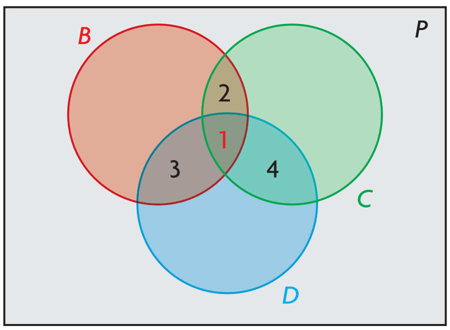
$$n(B) + n(C) + n(D) - n(B \cap C) - n(B \cap D) - n(C \cap D) + n(B \cap C \cap D) = n(B \cup C \cup D)$$

Each circle overlaps the other two circles. I determined the sum of the three circles using the Principle of Inclusion and Exclusion to deal with the overlapping areas.

$$\begin{aligned} 13 + 13 + 13 - (x + 2) - (x + 3) - (x + 4) + x &= 28 \\ 39 - x - 2 - x - 3 - x - 4 + x &= 28 \\ 30 - 2x &= 28 \\ -2x &= -2 \\ x &= 1 \end{aligned}$$

I knew that each circle represents 13 children, and there are 28 children in total. I solved for x.

One child has three different types of pets.



The area of the Venn diagram where all three circles overlap represents children who have all three pets. I revised my diagram.

b) Children with one pet $=$ Total number of children $-$ Children who have more than one pet

Children with one pet $= 28 - (1 + 2 + 3 + 4)$
Children with one pet $= 28 - 10$
Children with one pet $= 18$

I subtracted the number of children who have more than one pet from the total number of children.

Therefore, 18 children have only one type of pet.

Your Turn

Suppose that the first clue is changed so there are 24 children who have a dog, a cat, or a bird. All the other clues remain the same. Determine the number of children who have all three types of pet and the number of children who have only one type of pet.

EXAMPLE 2 Searching on the Internet

Hillary and Liam are working on a project for their World Issues class. They need to use the Internet to gather information about popular culture, especially social criticism on television. Liam suggests they search for popular culture using a search engine.

How can they refine their search to narrow down the number of hits?

Hillary's Solution

$U = \{$all sites on the Internet$\}$
$R = \{$sites containing the words *popular* and *culture*$\}$

I defined the universal set and the set that Liam suggested searching.

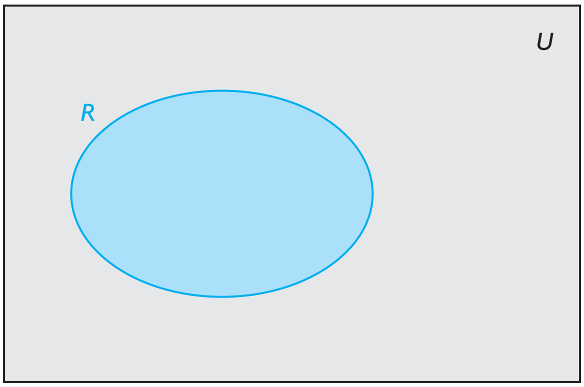
popular culture

About 186,000,000 results (0.08 seconds)

The results included all the sites with the words *popular* and *culture*, in any order and not necessarily together.

These are too many results for me to analyze.

I drew a Venn diagram to illustrate my search.



"popular culture"

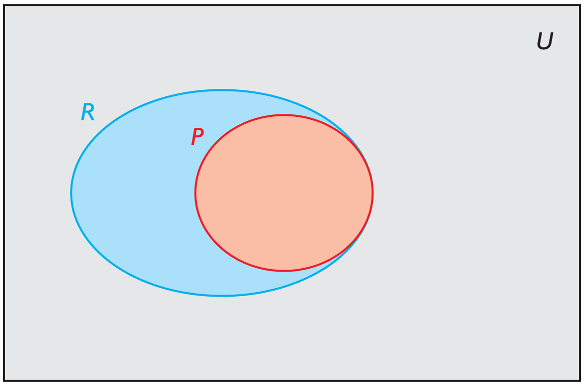
About 6,520,000 results (0.08 seconds)

I got fewer results, but still too many to analyze.

I refined my search using quotation marks, so the results included only sites with the exact phrase "*popular culture.*"

P = {sites containing the phrase "*popular culture*"}

I defined set P.



I revised my Venn diagram.

I did not draw the shapes to scale. The shapes just show the relationship between the two sets. Set P is a subset of set R.

The second search is a subset of the first search.

"popular culture" and television

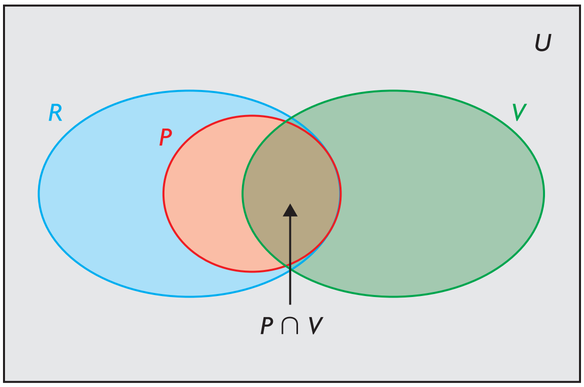
About 2,570,000 results (0.30 seconds)

I searched "*popular culture*" and *television* together. I used "and" so I could locate articles that contained both.

I got even fewer results, but still too many.

V = {all *television* sites}

I defined set V and revised my diagram.



The area of my diagram that contains both "*popular culture*" and *television* is the intersection of sets P and V.

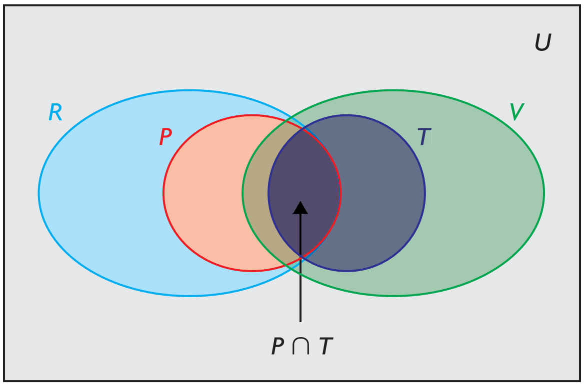
"popular culture" and "television shows"

About 105,000 results (0.15 seconds)

Some sites were not about television shows, so I refined my search. I put quotation marks around "*television shows*" to search for the exact phrase.

It's getting better.

$T = \{$"*television shows*" sites$\}$



I defined set T and revised my diagram. Set T is a subset of set V.

The area of my diagram that contains both "*popular culture*" and "*television shows*" is the intersection of sets P and T.

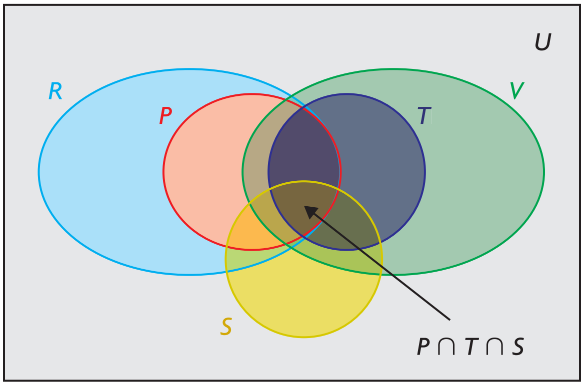
"popular culture" and "television shows" and "social criticism"

About 962 results (0.20 seconds)

I included "*social criticism*" in quotation marks to narrow down the number of hits.

$S = \{$"*social criticism*" sites$\}$

I defined set S and revised my diagram. Set S intersects the other sets.



The area of my diagram that contains "*popular culture*," "*television shows*," and "*social criticism*" is the intersection of sets P, T, and S.

I had about 962 results. I can deal with this number of results.

I narrowed my search from 186 million hits to only 962 hits.

$P \cap T \cap S = \{$popular culture and social criticism on television shows$\}$

$n(P \cap T \cap S) = 962$

Since my final search included all three criteria, it consisted of the intersection of all three sets.

Your Turn

Liam knows that animated television shows often use humour to comment on serious social issues. How would including the word *animated* in an Internet search affect the number of hits? Explain.

EXAMPLE 3 Correcting errors that involve sets

Shannon's high school starts a campaign to encourage students to use "green" transportation for travelling to and from school. At the end of the first semester, Shannon's class surveys the 750 students in the school to see if the campaign is working. They obtain these results:

- 370 students use public transit.
- 100 students cycle and use public transit.
- 80 students walk and use public transit.
- 35 students walk and cycle.
- 20 students walk, cycle, and use public transit.
- 445 students cycle or use public transit.
- 265 students walk or cycle.

How many students use green transportation for travelling to and from school?

Alaina solved this problem as shown below, but she made some errors. What errors did she make? Determine the correct solution.

Alaina's Solution

445 students cycle or use public transit.
370 use public transit.

Therefore, 445 − 370 or 75 students cycle but do not walk or use public transit.

I knew how many students use public transit or cycle and how many use public transit, so I subtracted to determine the number who only cycle.

265 walk or cycle.
75 students just cycle.

Therefore, 265 − 75 or 190 students only walk.

I knew how many students cycle and how many walk or cycle. I used this information to determine the number of students who only walk.

370 students use public transit.
75 students only cycle.
190 students only walk.
100 students cycle and use public transit.
80 students walk and use public transit.
35 students walk and cycle.
20 students use all three methods of green transportation.

I listed the number of students in each category.

Therefore, 370 + 75 + 190 + 100 + 80 + 35 + 20 or 870 students use green transportation.

I added the numbers in each category. I knew that my answer is wrong because there are only 750 students in the school.

Alberto's Solution

When determining how many students cycle, Alaina did not account for the number of students who both cycle and use public transit, or for the number of students who use all three methods of transportation. She made the same mistake when determining how many students walk.

I examined Alaina's solution and identified the errors she made.

Let U represent the universal set:

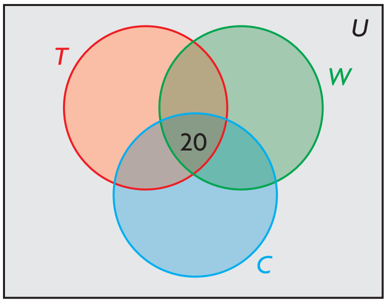
U = {students who attend Shannon's school}

T = {students who use public transit}

W = {students who walk}

C = {students who cycle}

I defined the sets in this situation.



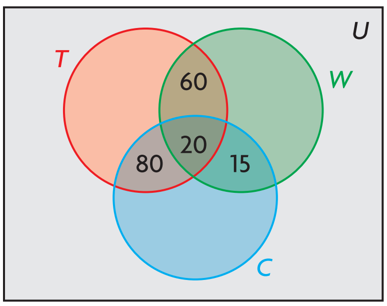
I drew a Venn diagram that showed the number of elements in each region.

I knew that 20 students use all three methods of transportation. I entered 20 where the three circles intersect.

The number who walk and use public transit is 80. Of those students, 20 have already been counted (the number who do both of those things and also cycle). That leaves $80 - 20$ or 60 to go in the other region in the intersection of T and W.

The number who walk and cycle is 35. Of those students, 20 have already been counted. That leaves $35 - 20$ or 15 to go in the other region of the intersection of C and W.

The number who cycle and use public transit is 100. Of those students, 20 have already been counted. That leaves $100 - 20$ or 80 to go in the other part of the intersection of T and C.



The number of students who use public transit is 370. Of those, $60 + 20 + 80$ or 160 have already been counted. The number who use only public transit is $370 - 160$ or 210.

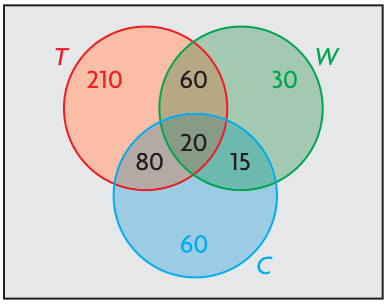
I used the same reasoning to determine the number of students who only use public transit, or walk, or only cycle.

The number of students who cycle or use public transit is 445. Of those, 370 use public transit. The number who cycle but don't use public transit is $445 - 370$ or 75.

I knew that "or" means the union of two sets.

Of the 75 students who cycle but don't use public transit, 15 have already been counted. The number who only cycle is $75 - 15$ or 60.

The number of students who walk or cycle is 265. Of those, $60 + 80 + 15 + 20 + 60$ or 235 have already been counted. The number who only walk is $265 - 235$ or 30.



$$n(T) = 210 + 60 + 20 + 80$$
$$n(T) = 370 \quad \checkmark$$

$$n(C \cup T) = 60 + 15 + 370$$
$$n(C \cup T) = 445 \quad \checkmark$$

$$n(W \cup C) = 60 + 20 + 15 + 30 + 80 + 60$$
$$n(W \cup C) = 265 \quad \checkmark$$

I checked my Venn diagram by verifying the number of students who use public transit, who cycle or use public transit, and who walk or cycle.

The numbers in my Venn diagram are reasonable.

Your Turn

How many students use exactly one method of green transportation?

EXAMPLE 4 Winning a game

Star is playing a game that involves sets. She is using the nine cards shown, which have three different attributes: shape, colour, and number of shapes. There are three shapes (triangle, square, and circle), three colours (red, blue, and green), and three numbers of shapes (one, two, and three).

To win, Star must create four sets, using three cards in each set from the nine cards shown. Each card may be used more than once in a set.

What sets can Star make to win the game?

Rules for Creating Sets

Sets of three cards must agree with each other or disagree with each other with respect to each attribute. Three cards form a set if

- the cards display the same number of figures or each displays a different number of figures, **and**
- the figures are all the same shape or three different shapes, **and**
- the figures are the same colour or three different colours.

For example:

Set 1: 

Set 2: 

Not a set: 

Star's Solution

B = {blue cards} $n(B) = 5$
G = {green cards} $n(G) = 2$
R = {red cards} $n(R) = 2$

First, I examined the attribute colour.

I do not have enough green cards or red cards to make a set of three with the same colour.

I can make a set of blue cards with different numbers and shapes:
1 blue square, 2 blue triangles, and 3 blue circles.

I examined the blue cards.
One card has one shape.
Three cards have two shapes.
One card has three shapes.
Two cards have squares, two have circles, and one has triangles.

S = {square cards} $n(S) = 2$

I examined the attribute shape. Only two cards have squares, so I cannot make a set with all squares.

I cannot make a set where all the shapes on the cards are squares.

T = {triangle cards} $n(T) = 4$
T = {1 red, 2 blue, 2 green, 3 green}

Four cards have triangles. There are three different colours and numbers of triangles on these cards. By the rules, all the colours and all the numbers need to be different.

I can make a set with the same triangle shape:
1 red triangle, 2 blue triangles, and 3 green triangles.

C = {circle cards} $n(C) = 3$
C = {2 blue, 2 red, 3 blue}

There are only two different numbers of circles on the circle cards, and only two different colours. To make a set, the colours must be all the same or all different.

I cannot make a set where all the shapes on the cards are circles.

O = {1-shape cards} $n(O) = 2$
W = {2-shape cards} $n(W) = 5$
H = {3-shape cards} $n(H) = 2$

I examined the attribute number. I knew that I could not make a set with the 1- or 3-shape cards since I have only two of them.

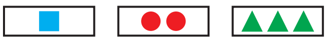
I cannot make a set where the number of shapes on each card is one or three.

To make a set with the same number, I used three different shapes and three different colours.

I can make a set with the same number, 2:
2 blue squares, 2 green triangles, and 2 red circles.

Finally, I looked for a set in which all three attributes were different.

I can make a set where all the attributes are different.
1 blue square, 2 red circles, and 3 green triangles.

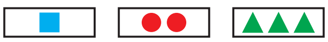


I made four sets of three cards.

Set 1: 

Set 2: 

Set 3: 

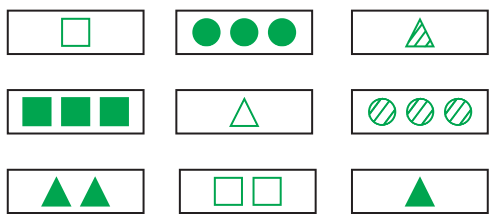
Set 4: 

I won the game. My sets are:

Set 1: same colour, different shape, different number
Set 2: same shape, different colour, different number
Set 3: same number, different shape, different colour
Set 4: different shape, different colour, different number

Your Turn

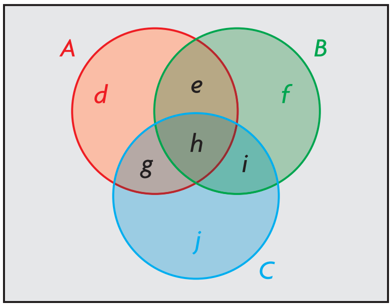
The following cards can be used for another game. Win this game by creating four sets of three cards, using the attributes of shape, number, and shading.



In Summary

Key Ideas

- Set theory is useful for solving many types of problems, including Internet searches, database queries, data analyses, games, and puzzles.
- To represent three intersecting sets with a Venn diagram, use three intersecting circles. For example, in the following Venn diagram,



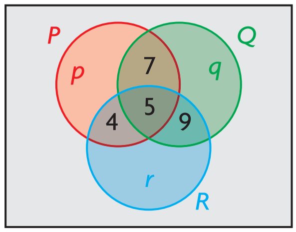
  - $A \cap B \cap C$ is represented by region h,
  - $A \cap B$ is represented by the union of regions e and h,
  - $A \cap C$ is represented by the union of regions g and h, and
  - $B \cap C$ is represented by the union of regions h and i.

  Each region of a Venn diagram contains elements that occur only in that particular region.
- You can use the Principle of Inclusion and Exclusion to determine the number of elements in the union of three sets:
  $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

Need to Know

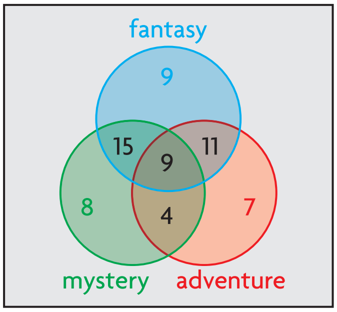
- You can use concepts related to sets to search for websites on the Internet:
  - Put an exact phrase in quotation marks.
  - Connect words or phrases with "and" to search for sites that contain both. The word "and" represents the intersection of two or more sets.
  - Connect words or phrases with "or" to search for sites that contain either one or the other, or both. The word "or" represents the union of two or more sets.
- When solving a puzzle or problem, it is often useful to visualize the problem. First identify which sets are defined by the context. Then identify how the sets overlap. Finally, identify regions of the overlaps that are of interest in the puzzle or problem. It is often advisable to consider how much is known about each region, and use the information about the region that is most known to deduce information about regions that are less well known. A systematic approach will result in answers that are easier to verify.

CHECK Your Understanding



1. The three circles in the Venn diagram (P, Q, and R) contain the same number of elements. Determine one set of values for p, q, and r.

2. The members of a book club read fantasy, mystery, and adventure books. The following Venn diagram shows the types of books that the members like:



Use the diagram to determine each amount below.

a) $n((F \cup M) \setminus A)$
b) $n((A \cup F) \setminus M)$
c) $n((F \cup A) \cup (F \cup M))$
d) $n(A \setminus F \setminus M)$

PRACTISING

3. Someone left a backpack full of school books on a transit bus. The only identification is the name "David Smith," so the bus driver takes the backpack to the public school board office. The staff search their database and learn that 56 students have this name. How can the staff narrow their search using search tools and other items in the backpack?

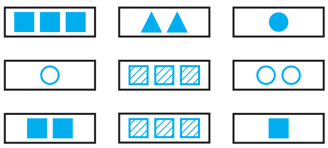
4. Jennifer is an optician. She is trying to decide whether she should offer a package deal to customers who buy glasses and contact lenses. She hires a survey company to research consumer preferences. A survey of 641 people provides the following information:
   - 83 wear contact lenses.
   - 442 wear glasses.
   - 167 do not use corrective lenses.

   What percent of Jennifer's customers might use a package deal? Use set notation in your answer.

5. Jacques is planning a winter ski holiday in the Canadian Rockies. Give four words or phrases that Jacques might use to search for information on the Internet. Use set theory to explain how quotation marks and the word "and" could help him refine his search.

6. A total of 58 teens attended a sports camp to train in at least one of three sports: swimming, cycling, and running.
   - 35 trained in swimming, 32 trained in cycling, and 38 trained in running.
   - 9 trained in swimming and cycling, but not in running.
   - 11 trained in cycling and running, but not in swimming.
   - 13 trained in swimming and running, but not in cycling.

   A triathlon consists of swimming, cycling, and running. How many teens might be training for the upcoming triathlon?



7. These nine attribute cards have three different shapes, numbers, and shadings (clear, striped, or solid).
   Determine three sets, with three cards in each set. Each set of three cards must have
   - the same number or three different numbers, and
   - the same shape or three different shapes, and
   - the same shading or three different shadings.

   All the cards can be used more than once.

8. Travis wants to buy a specific model of car. He goes into a car dealership in Medicine Hat, but the dealer does not have this model. The dealer searches the database and discovers that a Camrose dealership has four models, a Red Deer dealership has six models, a Sherwood Park dealership has five models, and a Lethbridge dealership has two models.
   a) What other attributes can the dealer use to narrow down the choices?
   b) How might the dealer prioritize the search?

9. John was asked to solve the following problem:
   240 students were surveyed to determine which restaurants they like.
   - 90 like Chicken and More.
   - 90 like Fast Pizza.
   - 90 like Gigantic Burger.
   - 37 like Chicken and More and Fast Pizza, but not Gigantic Burger.
   - 19 like Chicken and More and Gigantic Burger, but not Fast Pizza.
   - 11 like Fast Pizza and Gigantic Burger, but not Chicken and More.
   - 13 like all three restaurants.

   How many students do not like any of these restaurants?

   John solved the problem as follows:

   John's Solution:
   I added up the first six results of the survey and subtracted the number of students who ate at all three restaurants. Then I subtracted this value from the total number of students surveyed.

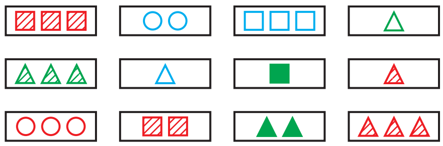
   $$90 + 90 + 90 + 37 + 19 + 11 - 13 = 324$$
   $$240 - 324 = -84$$

   This answer is not possible, so I knew that I made an error.

   What error did John make? What is the correct answer?

10. Wilson is searching online for information about local colleges and their athletics programs. He is interested in colleges in Edmonton or Calgary, but not universities.
   a) His first search term is *colleges*. How can he categorize colleges in Edmonton or Calgary?
   b) Since Wilson is interested in colleges and their athletics programs, should he use "and" or "or" to connect them?
   c) Should Wilson use "and" or "or" to search for one or the other city?
   d) To exclude universities, Wilson used the notation –*university*. The minus sign means "not." What might Wilson's search instructions look like?
   e) Try searching for the information that Wilson wants. What is the smallest number of hits you found?
   f) Represent your results in a Venn diagram.

11. These 12 cards have three different colours, shapes, numbers, and shadings.



Determine six sets of cards, with three cards in each set. Each set of three cards must have
- the same number or three different numbers, and
- the same shape or three different shapes, and
- the same colour or three different colours, and
- the same shading or three different shadings.

All the cards can be used more than once.

12. The cards in question 11 are part of a complete deck of cards. Determine the following amounts:
   a) $n(D)$, the total number of cards in the deck
   b) $n(T)$, the total number of triangle cards in the deck
   c) $n(G)$, the total number of green cards in the deck
   d) $n(S)$, the total number of cards with shading
   e) $n(T \cup G)$
   f) $n(G \cap S)$

13. A small web-hosting service specializes in websites involving outdoor activities.

- 35 sites involve boats: 20 of these sites deal with fishing boats and 25 deal with power boats.
- 21 sites involve fishing: these sites include all the sites that deal with fishing boats; 3 sites deal with fly fishing.

   a) How many sites from this service would appear in a search for *fishing boats*? Explain.
   b) Why might a search for *fishing* and *boats* turn up a different result than a search for "*fishing boats*"?
   c) If the only search word was *fishing*, how many results would not involve boats?

Closing

14. James searched for "*string bean*" on the Internet with quotation marks. Elinor searched for *string bean* without quotation marks. Did they get the same results? Explain, using set theory and a Venn diagram.

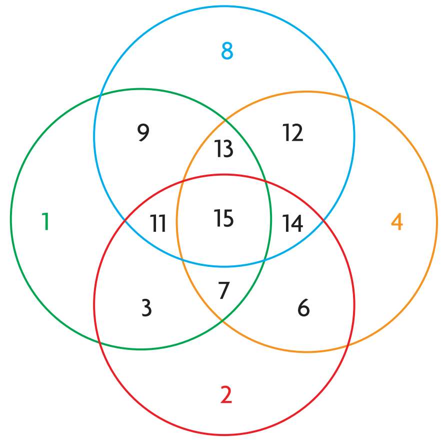
Extending

15. a) Four sets, A, B, C, and D, all intersect. Represent this using a Venn diagram. You do not need to limit the shapes to circles.

b) Number each area in your Venn diagram.

c) List the set combinations for each set. For example, the area that shows the intersection of only A, B, and C would be $(A \cap B \cap C) \setminus D$.

16. Explain why the following Venn diagram is not adequate to show four intersecting sets.



Math in Action

Relevant Hits

Have you noticed that many of the hits in an Internet search are not relevant to the topic you are searching?

Choose a topic that interests you. Decide on keywords to help you narrow your search.

Will your search involve the union of these words, the intersection of these words, or both? Are there any exact phrases you want to use, or do you just want hits that contain the words or variations of the words?

- Try searching with quotation marks, without quotation marks, and with the words "or" or "and." Adjust your search tools until you are satisfied with your results.
- Record the exact search you used, and explain why it worked. Show your results using a Venn diagram. For example, suppose that you want to adopt a dog. Your search might look like this:

  "golden retriever" or "Labrador retriever" and Alberta
- Does your browser's Advanced Search feature give you any more ideas on how to narrow your search?

3.5 Conditional Statements and Their Converse

GOAL

Understand and interpret conditional statements.

EXPLORE…

Consider the following two statements:

- If Briony is texting, then she is using a cellphone.
- If Briony is using a cellphone, then she is texting.

How do these statements relate to each other? Are they both true?

LEARN ABOUT the Math

James and Gregory like to play soccer, regardless of the weather. Their coach made this **conditional statement** about today's practice:
"If it is raining outside, then we practise indoors."

conditional statement

An "if–then" statement; for example, "If it is Monday, then it is a school day."

? When will the coach's conditional statement be true, and when will it be false?

EXAMPLE 1 Verifying a conditional statement

Verify when the coach's conditional statement is true or false.

James's Solution: Using reasoning and a truth table

Hypothesis: "It is raining outside."
Conclusion: "We practise indoors."

I identified the **hypothesis** by writing the statement that followed "If."

I identified the **conclusion** by writing the statement that followed "then."

Each of these statements is either true or false, so to verify this conditional statement, I need to consider four cases.

hypothesis

An assumption; for example, in the statement "If it is Monday, then it is a school day," the hypothesis is "It is Monday."

conclusion

The result of a hypothesis; for example, in the statement "If it is Monday, then it is a school day," the conclusion is "it is a school day."

Case 1: The hypothesis is true and the conclusion is true. *It rains outside, and we practise indoors.*
When the hypothesis and conclusion are both true, a conditional statement is true.

The coach kept his promise, so the coach's conditional statement is true.

Case 2: The hypothesis is false, and the conclusion is false. *It does not rain outside, and we practise outdoors.*
When the hypothesis and conclusion are both false, a conditional statement is true.

Since it does not rain outside, the coach is not obligated to keep his promise. We can practise either indoors or outdoors, so the coach's conditional statement is true.

Case 3: The hypothesis is false, and the conclusion is true. *It does not rain outside, and we practise indoors.*
When the hypothesis is false and the conclusion is true, a conditional statement is true.

Since it does not rain outside, the coach is not obligated to keep his promise. We can practise either indoors or outdoors, so the coach's conditional statement is true.

Case 4: The hypothesis is true, and the conclusion is false. *It rains outside, and we practise outdoors.*
When the hypothesis is true and the conclusion is false, a conditional statement is false.

The coach has broken his promise. We should be practising indoors, not outdoors. This **counterexample** shows that the coach's conditional statement is false.

counterexample

An example that disproves a statement; for example, "If it is Monday, then it is a school day" is disproved by the counterexample that there is no school on Thanksgiving Monday. Only one counterexample is needed to disprove a statement.

Let p represent the hypothesis:
It is raining outside.
Let q represent the conclusion:
We practise indoors.

I decided to create a truth table to summarize my observations. I began by representing the hypothesis and conclusion with variables.

p	q	$p \Rightarrow q$
T	T	T
F	F	T
F	T	T
T	F	F

I entered my analysis from each of the four cases above in a truth table.

From the truth table I can see that the only time a conditional statement will be false is when the hypothesis is true and the conclusion is false. This means that when I assume that the hypothesis in a conditional statement is true, I can determine if the conditional statement is true or false based only on whether the conclusion is true or false.

When the hypothesis is false, regardless of whether the conclusion is true or false, the conditional statement is true. When the hypothesis is true and the conclusion is true, the conditional statement is true.

Gregory's Solution: Using reasoning and a Venn diagram

U = {The universal set of all practice times}
P = {The set of all times when it is raining}
Q = {The set of all times when we practise indoors}

I defined the sets I could use in this situation.

Let p represent the hypothesis: *It is raining outside.*
Let q represent the conclusion: *We practise indoors.*

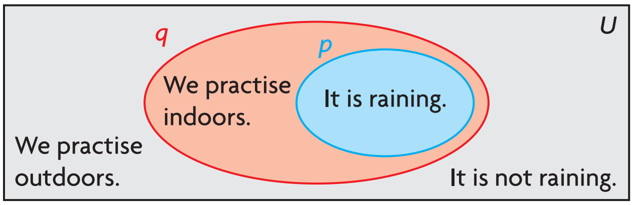
I defined variables to represent the hypothesis and the conclusion.

Suppose t is a practice time; that is, an element of the universal set of all times, U. It follows that:
$P \subset Q$

If $t \in P$, then it is raining at time t, and so $p \Rightarrow q$ means that $t \in Q$. Thus $P \subset Q$.

If $t \notin P$, then it is not raining at time t. In this case, $p \Rightarrow q$ gives no information about whether t is an element of Q.

To represent a conditional statement using a Venn diagram, I drew a small circle for the hypothesis within a large circle for the conclusion.



A Venn diagram supports my reasoning. Clearly the relationship of the two circles is correct. If the conditional statement is true, t is a time when it rains outside, so t belongs inside the small circle. In this case, we practise indoors, so t must also belong inside the larger circle. This can only happen if P is a subset of Q.

Further, if it is not raining outside, then t belongs outside the small circle and may or may not lie inside the large circle. (Both are possible, since we can practise either indoors or outdoors when it is not raining outside.)

From the Venn diagram, I can see that the conditional statement will be true when

- the hypothesis is true, and the conclusion is true.
- the hypothesis is false.

From the Venn diagram, I can see that the only time a conditional statement will be false is when the hypothesis is true and the conclusion is false.

A time t can never lie inside P and outside Q at the same time, which would have to happen if the conditional statement is false.

Reflecting

A. Examine James's truth table.
   i) What do you notice about the conditional statement when the hypothesis is false?
   ii) What do you notice about the statement when the conclusion is true?
   iii) What do you notice about the statement when the hypothesis is true and the conclusion is false?

B. How can you use a Venn diagram to decide if a conditional statement is true or false?

Communication | *Notation*

$p \Rightarrow q$ is notation for "If p, then q."
$p \Rightarrow q$ is read as "p implies q."

APPLY the Math

EXAMPLE 2 Determining if the converse of a conditional statement is true

Recall the coach's conditional statement: "If it is raining outside, then we will practise indoors."

Is the **converse** of this conditional statement true or false? Justify your decision.

converse

A conditional statement in which the hypothesis and the conclusion are switched; for example, the converse of "If it is Monday, then it is a school day" is "If it is a school day, then it is Monday."

Gregory's Solution

Converse: "If we practise indoors, then it is raining outside."

I wrote the converse by switching the hypothesis and the conclusion.

We could be practising indoors because the soccer field is under repair, not because it is raining.

I assumed that the hypothesis "we practise indoors" is true. I needed to decide if the conclusion "it is raining outside" will always be true.

This counterexample proves that the converse of the coach's statement is false.

I found a counterexample, so the conclusion is false.

Your Turn

Consider the following conditional statement: If a whole number is divisible by 10, then its last digit is 0.

a) Is this conditional statement true or false? Explain.

b) Is the converse of this conditional statement true or false? Explain.

EXAMPLE 3 Determining if a statement is biconditional

Sayyna told her friend Pipaluk, "If you are north of latitude 60° N, you can experience over 18.8 h of daylight on June 21."

a) Is Sayyna's statement true?

b) Write the converse. Is it true?

c) Is Sayyna's statement **biconditional**?

biconditional

A conditional statement whose converse is also true; in logic notation, a biconditional statement is written as "p if and only if q." For example, the statement "If a number is even, then it is divisible by 2" is true. The converse, "If a number is divisible by 2, then it is even," is also true. The biconditional statement is "A number is even if and only if it is divisible by 2."

Pipaluk's Solution: Using reasoning

a) The southern borders of Yukon, Northwest Territories, and Nunavut lie along the latitude 60° N.

I determined the hours of daylight at this latitude on June 21.

On an Hours of Daylight map, the maximum hours of daylight at this latitude is 18 h 53 min.

Let p represent the hypothesis: *You are north of latitude 60° N.*

Let q represent the conclusion: *You can experience over 18.8 h of daylight on June 21.*

p	q	$p \Rightarrow q$
T	T	T

Sayyna's statement "If you are north of latitude 60° N, you can experience over 18.8 h of daylight on June 21" is true.

I assumed that p is true. I need to decide if the conclusion, q, that follows is true.

If you are north of latitude 60° N, you can experience over 18.8 h of daylight on June 21. The conclusion is true.

When the hypothesis is true and the conclusion is true, the conditional statement is true.

b) The converse of Sayyna's statement is "If you can experience over 18.8 h of daylight on June 21, then you are north of latitude 60° N."

To write the converse, I switched the order of the hypothesis and the conclusion.

Based on my original definitions of the variables:

q is now the hypothesis: *You can experience over 18.8 h of daylight on June 21.*

p is now the conclusion: *You are north of latitude 60° N.*

q	p	$q \Rightarrow p$
T	T	T

I assumed that the new hypothesis is true. You experience over 18.8 h of daylight. This means you are north of latitude 60° N. The new conclusion is true.

When the hypothesis is true and the conclusion is true, the conditional statement is true.

The converse of Sayyna' statement, "If you experience over 18.8 h of daylight on June 21, then you are north of latitude 60° N," is also true.

Since Sayyna's conditional statement and its converse are true, the statement is biconditional.

c) $p \Rightarrow q$ is true.
$q \Rightarrow p$ is true.
Therefore, $p \Leftrightarrow q$ is true.
Sayyna's statement is biconditional.

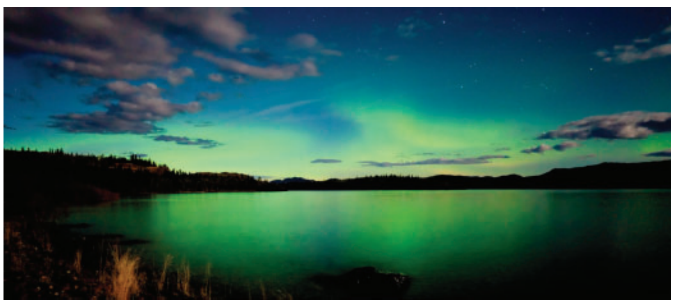
Communication | *Notation*

$p \Leftrightarrow q$ is notation for "p if and only if q."
This means that both the conditional statement and its converse are true statements.

Your Turn

Sanela made this conditional statement: "If you live in Nunavut, you will experience days with less than 5.9 h of daylight."

a) Is Sanela's statement true?
b) Write the converse. Is it true?
c) Is Sanela's statement biconditional?



EXAMPLE 4 Writing conditional statements

"A person who cannot distinguish between certain colours is colour blind."

a) Write this sentence as a conditional statement in "if p, then q" form.
b) Write the converse of your statement.
c) Is your statement biconditional? Explain.

Emile's Solution

a) If p, then q.
If a person cannot distinguish between certain colours, then that person is colour blind.

I wrote the first part of the sentence as p, and the second part as q.

b) If q, then p.
If a person is colour blind, then that person cannot distinguish between certain colours.

I wrote the converse by switching the hypothesis and the conclusion.

c) The first statement is true.
The converse is also true.

I looked up the definition of colour blindness in the dictionary.

Since both the conditional statement and its converse are true, the statement is biconditional.
I can write the statement as:
"A person is colour blind if and only if that person cannot distinguish between certain colours."

A person who cannot distinguish between certain colours is colour blind, and a person who is colour blind cannot distinguish between certain colours.

Your Turn

A square has four right angles.
a) Write this sentence as a conditional statement in "if p, then q" form.
b) Write the converse of your statement.
c) Is your statement biconditional? Explain your reasoning.

EXAMPLE 5 Verifying a biconditional statement

Reid stated the following biconditional statement: "A quadrilateral is a square if and only if all of its sides are equal." Is Reid's biconditional statement true? Explain.

Emanuella's Solution

Conditional statement: "If a quadrilateral is a square, then all of its sides are equal."

I wrote Reid's statement as a conditional statement.

This conditional statement is true.

If I assume that a quadrilateral is a square, then the conclusion that it has four equal sides is also true. Since the conclusion is true when the hypothesis is true, the conditional statement is true.

"If all the sides of a quadrilateral are equal, then it is a square."

I wrote the converse.

A counterexample is a rhombus. Therefore, the converse is false.

If I assume that all the sides of a quadrilateral are equal, then the conclusion that the shape is a square is false. A rhombus has four equal sides, but a rhombus is only a square if all four angles measure 90°. Since the conclusion is false when the hypothesis is true, the conditional statement is false.

Since the converse of the conditional statement is false, the biconditional statement is false.

Your Turn

Meredith wrote the following biconditional statement: "A quadrilateral is a parallelogram if and only if its opposite sides are parallel." Is Meredith's biconditional statement true? Explain.

EXAMPLE 6 Making a decision based on a conditional financial statement

Brian and Anna want to buy a house. They have determined that they can afford to make a mortgage payment of $1400 each month.

a) If the current interest rate is 4%, the term of the mortgage is 5 years, and the mortgage will be amortized over 25 years, then what is the maximum mortgage they can afford?

b) If the interest rate doubled in 5 years, then how would this increase affect the maximum mortgage they might consider today? What advice can you give?

Bill's Solution: Using technology

a) The number of payments is $25 \cdot 12$ or 300.
The interest rate is 4%.
The present value is unknown.
The payments are $1400.
The future value is $0.
The payment frequency is 12.
The compounding frequency is 2.

I used the finance application on my calculator to determine the maximum mortgage payment.

I knew that in Canada, all mortgage interest is compounded semi-annually.

Therefore, the present value is $266 149.47. The maximum mortgage they should consider is $266 149.47.

b) The number of payments is $25 \cdot 12$ or 300.
The interest rate is 8%.
The present value is unknown.
The payment is \$1400.
The future value is \$0.
The payment frequency is 12.
The compounding frequency is 2.

If the interest rate doubled, the mortgage might be too costly. I determined the maximum mortgage that they could afford today while keeping their monthly payment at \$1400. I used 8% as the interest rate over the entire 25-year period.

Therefore, the present value is \$183 434.92.

I determined the present value for a 25-year mortgage at 8%. Since they will have paid off some of the principal of their mortgage over the first 5 years at 4%, they will actually be able to carry a mortgage a little greater than \$183 434.92 if interest rates increase to 8% for the next 20 years. However, to be safe, they should not get a mortgage for any more than \$185 000.

To ensure that they can still afford the monthly payments on the house in 5 years, when the mortgage comes up for renewal, they should get a smaller mortgage. They should look for a less expensive house.

Your Turn

a) Generate your own "what if" question, dealing with the financial considerations of Anna and Brian purchasing a house. Make a decision regarding your question. Justify your decision.
b) Do you think that Bill's advice is logically justified? Explain.

In Summary

Key Ideas

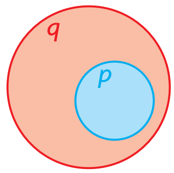
- A conditional statement consists of a hypothesis, p, and a conclusion, q. Different ways to write a conditional statement include the following:
  - If p, then q.
  - p implies q.
  - $p \Rightarrow q$
- To write the converse of a conditional statement, switch the hypothesis and the conclusion.

Need to Know

- A conditional statement is either true or false. A truth table for a conditional statement, $p \Rightarrow q$, can be set up as follows:

p	q	$p \Rightarrow q$
T	T	T
F	F	T
F	T	T
T	F	F

  A conditional statement is false only when the hypothesis is true and the conclusion is false. Otherwise, the conditional statement is true, even if the hypothesis is false.
- You can represent a conditional statement using a Venn diagram, with the inner oval representing the hypothesis and the outer oval representing the conclusion. The statement "p implies q" means that p is a subset of q.



- Only one counterexample is needed to show that a conditional statement is false.
- If a conditional statement and its converse are both true, you can combine them to create a biconditional statement using the phrase "if and only if."

CHECK Your Understanding

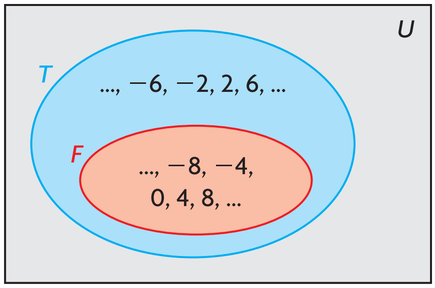
1. Consider the following conditional statement: "If I am swimming in the ocean, then I am swimming in salt water."

a) Write the hypothesis and the conclusion.

b) Is the conditional statement true? If it is false, provide a counterexample.

c) Write the converse. Is the converse true? If it is false, provide a counterexample.

2. Use the Venn diagram to answer the following questions.



   a) Consider the following conditional statement in relation to the Venn diagram: "If a number is divisible by 4, then it is divisible by 2." Is this statement true?
   b) Write the converse. Is the converse true?
   c) Determine a counterexample, if possible.

3. An equilateral triangle has three equal sides.
   a) Write this statement in "if p, then q" form.
   b) Write the converse of your conditional statement in part a).
   c) Is each statement true or false?
   d) Is the statement biconditional? Explain.

PRACTISING

4. A Spanish proverb says, "Since we cannot get what we like, let us like what we can get."
   a) Write the proverb in "if p, then q" form.
   b) What is the hypothesis? What is the conclusion?

5. Consider this conditional statement: "If a number is divisible by 5, then its final digit is a 0."
   a) Is this statement true?
   b) Write the converse.
   c) Is the converse true? Support your decision with a Venn diagram.

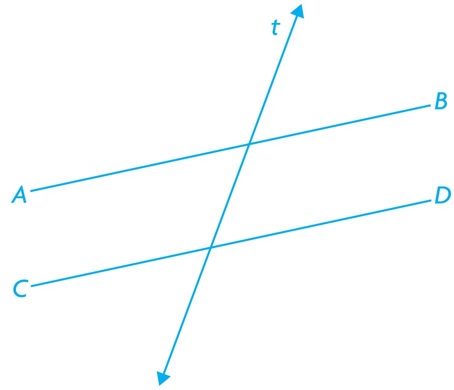
If you are in the capital of Canada, then you can visit Parliament.

6. Determine whether each statement is biconditional. If the statement is biconditional, rewrite it in biconditional form. If the statement is not biconditional, provide a counterexample.
   a) If you live in Canada, then you live in North America.
   b) If you live in the capital of Canada, then you live in Ottawa.

7. Use a truth table to determine whether the following statement is biconditional: If $\sqrt{x^2} = x$, then x is not negative.

8. Write each statement in "if p, then q" form. If the statement is biconditional, rewrite it in biconditional form. If the statement is not biconditional, provide a counterexample.
   a) A half-empty glass is half full.
   b) A rhombus has equal opposite angles.
   c) A repeating decimal can be expressed as a fraction.

9. A transversal, t, intersects line segments AB and CD. Consider this statement:
"The line segments AB and CD are parallel if and only if the alternate angles are equal."
   a) Write a conditional statement and its converse.
   b) Are the statements you wrote in part a) true or false? Explain how you know.
   c) Is the original statement true or false? Explain how you know.



10. Write the converse of each statement. Then determine if each statement is biconditional:
   a) If your pet barks, then it is a dog.
   b) If your pet is a dog, then it wags its tail.

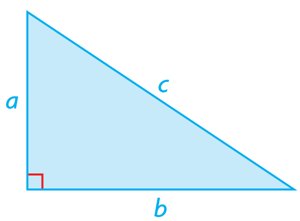
11. Is each statement true or false?
   a) If $x + y = z$, then $x = z - y$.
   b) If $p - q = r$, then $q + r = p$.

12. Consider the Sudoku puzzle. Write a conditional statement about the number that belongs in the shaded square. What conclusion can you make about where to place the numbers in that column?

	4			2	7			
	7		3	6	5		2	
2			9				7	
6		9						3
	1						9	
5	2					7		6
	3				4			8
	9		7	5	6		3	
			2	8			6	

13. For each statement below,
   i) write the statement in "if p, then q" form,
   ii) write the converse of the statement,
   iii) verify the statement and its converse, and
   iv) if the statement and its converse are both true, write the biconditional statement.

   a) A square has four right angles.
   b) For a right triangle, $a^2 + b^2 = c^2$.



   c) A trapezoid has two sides that are parallel.

14. a) Michelle and Marc are buying a house. If they require a \$250 000 mortgage, amortized for 25 years, at 6.5% compounded semi-annually, then:

i) Determine the amount of each mortgage payment if they make one payment a month.

ii) Determine the amount of each mortgage payment if they make semi-monthly payments.

b) If Michelle and Marc make payments twice a week, but pay the same total amount per month they would need to pay for semi-monthly payments, then how many mortgage payments would they make? What advice would you give Michelle and Marc?

Closing

15. a) Write two different statements in "if *p*, then *q*" form. One of these statements should be biconditional.

b) Represent each statement with a Venn diagram.

c) Explain how to tell whether the converse of each statement is true or false using the Venn diagram.

Extending

16. A cryptoquote is a code-breaking puzzle. In a cryptoquote, every word must contain a vowel and no letter ever represents itself. For example, consider the cryptoquote below:

```
AC   NG   CL   FCA   AC   NG
```

This cryptoquote is broken as follows:

```
To   be   or   not   to   be
```

To solve a cryptoquote, look for repeated letters, especially in two-letter words. A word that contains only one letter must be either "a" or "I."

a) Write an "if-then" statement for the two-letter words in a cryptoquote.

b) Solve the cryptoquote below. For this cryptoquote, assume that "J" represents "a."

```
KSQ QSCAXHBMV TD TY DKJD CSNSAU
CXXA QJTD J YTCLVX PSPXCD NXBSHX
YDJHDTCL DS TPZHSWX DKX QSHVA.
     – JCCX BHJCG
```

17. Suppose that Brian and Anna, from Example 6, decide to get a mortgage for the maximum amount they are allowed: \$266 149.47. If they decide to pay an additional \$250 per month to reduce the time required to pay off the mortgage, and the interest rate stays at 4%, then:

a) How many months will it take them to pay off the mortgage?

b) How much money will they save in total?

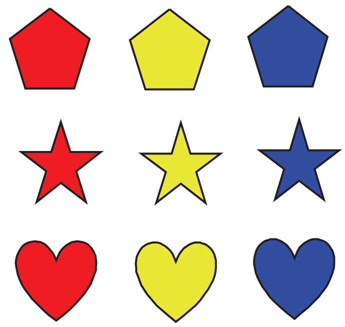
Applying Problem-Solving Strategies

Analyzing a Logic Puzzle

Logic can often help you solve a puzzle.

YOU WILL NEED

- coloured markers
- scissors
- Puzzle Shapes



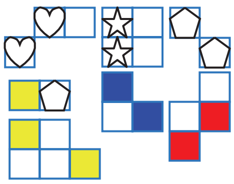
The Puzzle

A. In this activity, you will use all nine puzzle shapes and the given clues to form a three-by-three square. (You will need one of each puzzle shape in each colour.) The following hints will help you understand the clues:

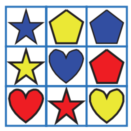
- A square with a solid colour means that the square must contain a shape of this colour.
- A hollow shape in a square means that the square must contain this shape.
- A solid symbol in a square means that square must contain this symbol.

Study Carol's solution to see how she used the clues she was given.

Carol's Clues

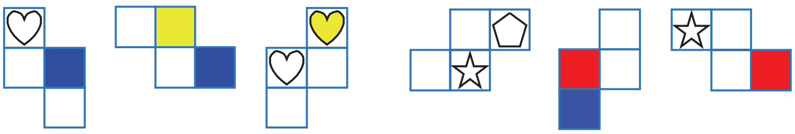


Carol's Solution



B. Use the nine puzzle shapes and the following clues to form a three-by-three square.

Your Clues



The Strategy

C. Describe the strategy you used to solve the puzzle in part B.

D. Create your own solution for a puzzle, and develop clues to solve the puzzle.

E. Give your clues to a partner. Can your partner solve your puzzle?

F. How could you adjust your clues to change the level of difficulty of your puzzle?

3.6 The Inverse and the Contrapositive of Conditional Statements

EXPLORE...

Mother Teresa said, "If you judge people, you have no time to love them."

- Decide whether this statement means the same as the following statement: "If you don't judge people, you have time to love them."

GOAL

Understand and interpret the contrapositive and inverse of a conditional statement.

INVESTIGATE the Math

Puneet's math teacher said, "If a polygon is a triangle, then it has three sides."

Puneet's geography teacher said, "If you live in Saskatoon, then you live in Saskatchewan."

She wonders what other statements she can write using this information.

? What other variations can Puneet write, and are they true?

A. Begin with the math teacher's statement. Is it true? Explain.

B. Write the converse of the math teacher's statement. Is it true? Explain.

C. Write the **inverse** of the math teacher's statement. Is it true? Explain.

D. Write the **contrapositive** of the math teacher's statement. Is it true? Explain.

E. Repeat parts A to D using the geography teacher's statement.

inverse

A statement that is formed by negating both the hypothesis and the conclusion of a conditional statement; for example, for the statement "If a number is even, then it is divisible by 2," the inverse is "If a number is **not** even, then it is **not** divisible by 2."

contrapositive

A statement that is formed by negating both the hypothesis and the conclusion of the **converse** of a conditional statement; for example, for the statement "If a number is even, then it is divisible by 2," the contrapositive is "If a number is **not** divisible by 2, then it is **not** even."

Reflecting

F. If you are given a conditional statement that you know is true, can you predict whether

i) the converse is true?

ii) the inverse is true?

iii) the contrapositive is true?

G. Test your conjectures from part F using this true statement: "If a quadrilateral is a rectangle, then it is a parallelogram."

H. Examine the statements you wrote for the inverse and converse for each of the conditional statements given by Puneet's teachers. What do you notice?

APPLY the Math

EXAMPLE 1 Verifying the inverse and contrapositive of a conditional statement

Consider the following conditional statement: "If today is February 29, then this year is a leap year."

a) Verify the statement, or disprove it with a counterexample.
b) Verify the inverse, or disprove it with a counterexample.
c) Verify the contrapositive, or disprove it with a counterexample.

Maggie's Solution: Using a truth table

a) "If today is February 29, then this year is a leap year."

Hypothesis (p): *Today is February 29.*
Conclusion (q): *This year is a leap year.*
Conditional statement: If p, then q.

I examined the conditional statement.

I determined the hypothesis, p, and the conclusion, q.

If I assume that the hypothesis, p, is true, then the conditional statement is true only when the conclusion, q, is also true.

p	q	$p \Rightarrow q$
T	T	T

Assuming that today is February 29 means that the hypothesis is true. Therefore, the conclusion is also true since February has 28 days, unless it is a leap year. Then February has 29 days.

This conditional statement is true.

b) Inverse: "If today is not February 29, then this year is not a leap year."

To write the inverse, I negated both the hypothesis and the conclusion.

Hypothesis ($\neg p$): *Today is not February 29.*
Conclusion ($\neg q$): *This year is a not a leap year.*
If $\neg p$, then $\neg q$.

$\neg p$	$\neg q$	$\neg p \Rightarrow \neg q$
T	F	F

Assuming that it is not February 29 means that the hypothesis is true. Therefore, the conclusion is false. For example, today could be January 5, 2012, but 2012 is a leap year. This counterexample shows that the inverse is false.

The inverse is false.

Communication | *Notation*

In logic notation, the inverse of "if p, then q" is written as "If $\neg p$, then $\neg q$."

c) Contrapositive: "If this year is not a leap year, then today is not February 29."

Hypothesis ($\neg q$): *This year is a not a leap year.*
Conclusion ($\neg p$): *Today is not February 29.*
If $\neg q$, then $\neg p$.

To write the contrapositive, I switched the hypothesis and the conclusion in the inverse.

Communication Notation

In logic notation, the contrapositive of "if p, then q" is written as "If $\neg q$, then $\neg p$."

$\neg q$	$\neg p$	$\neg q \Rightarrow \neg p$
T	T	T

The contrapositive is true.

Assuming that this year is not a leap year means that the hypothesis is true. Therefore, the conclusion is true since February 29 does not occur in any year but a leap year.

Your Turn

Consider the following conditional statement: "If this year is a leap year, then February has 29 days." Determine if the statement, its inverse, and its contrapositive are true or false.

EXAMPLE 2 Examining the relationship between a conditional statement and its contrapositive

Consider the following conditional statement: "If a number is a multiple of 10, then it is a multiple of 5."

a) Write the contrapositive of this statement.
b) Verify that the conditional and contrapositive statements are both true.

Bailey's Solution: Using reasoning

a) Conditional statement: "If a number is a multiple of 10, then it is a multiple of 5."

Contrapositive: "If a number is not a multiple of 5, then it is not a multiple of 10."

I wrote the contrapositive by negating both parts of the converse.
This is the same as negating the hypothesis and the conclusion, then switching their positions in the statement.

b) If n is a multiple of 10, then $n = 10m, m \in I$.

If I assume that a number is a multiple of 10, then the conclusion that the number is a multiple of 5 is true.

Since 10 is a factor of $10m$, n is divisible by 10.
Also, $n = 5(2m), m \in I$.
Since 5 is a factor of $5(2m)$, 5 is a factor of n.
The conditional statement is true.

Contrapositive: "If a number is not a multiple of 5, then it is not a multiple of 10."

The contrapositive is true.

The conditional and the contrapositive statements are both true.

Next, I considered the contrapositive.

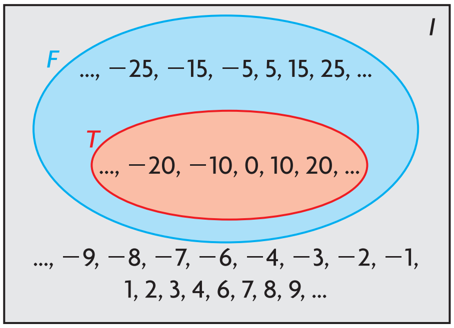
If I assume that a number is not a multiple of 5, then the conclusion that it is not a multiple of 10 is true.

Since all multiples of 10 are also multiples of 5, a number that is not a multiple of 5 cannot be a multiple of 10. The contrapositive is true.

Briony's Solution: Using a Venn diagram

a) $I = \{n \in \mathrm{I}\}$
$F = \{5n, n \in \mathrm{I}\}$
$T = \{10n, n \in \mathrm{I}\}$

I defined the universal set *I* of integers, set *F*, the multiples of 5, and set *T*, the multiples of 10.



I drew a Venn diagram to show how sets *I*, *F*, and *T* are related.

Conditional statement: "If a number is a multiple of 10, then it is a multiple of 5."

Contrapositive: "If a number is not a multiple of 5, then it is not a multiple of 10."

b) $T \subset F$
Therefore, the conditional statement is true.

I examined my Venn diagram.

The conditional statement shows that any number in subset *T* is in set *F*.

Any number that is not a multiple of 5 lies in F'.
$T \not\subset F'$
Therefore, the contrapositive is also true.

My Venn diagram shows that if a number is in F', it cannot be also in *T*. Therefore, F' and *T* are disjoint sets.

The conditional and contrapositive statements are both true.

I can represent both statements with the same Venn diagram.

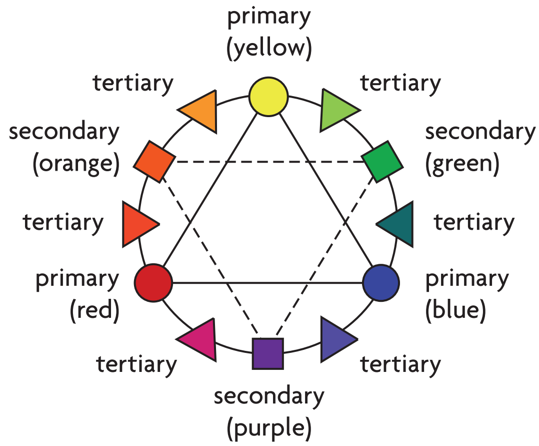
Your Turn

You are given a conditional statement that you know is true. What can you conclude about the contrapositive?

EXAMPLE 3 Examining the relationship between the converse and inverse of a conditional statement

Arizona is studying the colour wheel in art class. She observes the following: "If a colour is red, yellow, or blue, then it is a primary colour."

a) Write the converse of this statement.
b) Write the inverse of this statement.
c) Verify that the converse and the inverse are both true.
d) Is Arizona's statement biconditional? Explain.



John's Solution: Analyzing statements

a) Conditional statement: "If a colour is red, yellow, or blue, then it is a primary colour."

Converse: "If a colour is a primary colour, then it is red, yellow, or blue."

I wrote the converse by switching the hypothesis and the conclusion in Arizona's statement.

b) Inverse: "If a colour is not red, yellow, or blue, then it is not a primary colour."

I wrote the inverse by negating both the hypothesis and the conclusion in Arizona's statement.

c) The converse is true.

I assumed that a colour was a primary colour, so the hypothesis is true. The only primary colours are red, yellow, and blue, so the conclusion is true. When the hypothesis and conclusion are both true, the conditional statement is true.

The inverse is true.
The converse and inverse are both true.

I assumed that a colour was not red, yellow, or blue, so the hypothesis is true. The only primary colours are red, yellow, and blue. If a colour is not one of these three colours, it is not a primary colour, so the conclusion is true. When the hypothesis and conclusion are both true, the conditional statement is true.

d) A colour is red, yellow, or blue if and only if it is primary. Therefore, the statement is biconditional.

Since red, yellow, and blue are the only primary colours, the statement is biconditional.

This makes sense, since the original statement and its converse are both true.

Arizona's Solution: Using a Venn diagram

a) Converse: "If a colour is a primary colour, then it is red, yellow, or blue."

b) Inverse: "If a colour is not red, yellow, or blue, then it is not a primary colour."

C = {all colours on the colour wheel}

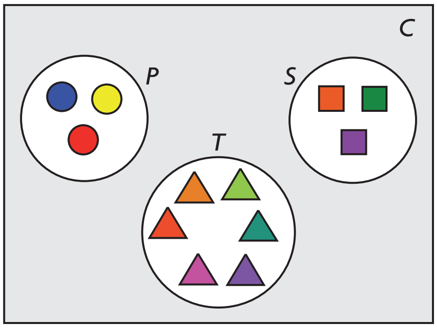
P = {red, yellow, blue}

S = {green, orange, purple}

T = {red-orange, yellow-orange, yellow-green, blue-green, blue-violet, red-violet}

> To write the converse, I switched the hypothesis and the conclusion.
>
> To write the inverse, I negated them.

> I defined the universal set C, then grouped and defined the colours on the wheel in three sets.



> I drew a Venn diagram to show how the sets are related. In this case, the three sets of colours on the colour wheel are disjoint sets.

c) Red, yellow, and blue are in the primary colour circle. There are no other colours in this circle.
Therefore, the converse is true.

All other colours, which are not primary colours, are either in set S or set T.
Therefore, the inverse is true.

The converse and the inverse are both true.

> I examined my Venn diagram.
>
> The converse shows that only red, yellow, and blue are primary colours.

> The inverse shows that if a colour is not red, yellow, or blue, it is not a primary colour.

> I can represent both statements with the same Venn diagram.

d) A colour is red, yellow, or blue if and only if it is primary. Therefore, the statement is biconditional.

> Since red, yellow, and blue are the only primary colours, the statement is biconditional.

Your Turn

You are given the converse of a conditional statement that you know is true. What can you conclude about the inverse?

In Summary

Key Ideas

- You form the inverse of a conditional statement by negating the hypothesis and the conclusion.
- You form the contrapositive of a conditional statement by exchanging and negating the hypothesis and the conclusion.

Need to Know

- If a conditional statement is true, then its contrapositive is true, and vice versa.
- If the inverse of a conditional statement is true, then the converse of the statement is also true, and vice versa.

CHECK Your Understanding

1. Write the converse, inverse, and contrapositive of each conditional statement.

a) If you find success before work, then you are looking in a dictionary.

b) If you are over 16, then you can drive.

c) If a quadrilateral is a square, then its diagonals are perpendicular.

d) If n is a natural number, then $2n$ is an even number.

2. Consider the following conditional statement: "If an animal has a long neck, then it is a giraffe."

a) Write the converse and the contrapositive of this statement.

b) Are the conditional and contrapositive statements both true? Explain.

3. Consider this statement: "If a polygon has five sides, then it is a pentagon."

a) Write the converse and the inverse.

b) Are the converse and the inverse both true? Explain.

4. Jeb claims that this statement is true: If $x^2 = 25$, then $x = 5$.

a) Do you agree or disagree with Jeb? Explain.

b) Is the converse true? Explain.

c) Is the inverse true? Explain.

d) Is the contrapositive true? Explain.

PRACTISING

5. For each conditional statement below,

i) determine if it is true,
ii) write the converse and determine if it is true,
iii) write the inverse and determine if it is true, and
iv) write the contrapositive and determine if it is true.

If any statement is false, provide a counterexample.

a) If you are in Hay River, then you are in the Northwest Territories.
b) If a puppy is male, then it is not female.
c) If the Edmonton Eskimos won every game this season, then they would be number 1 in the West.
d) If an integer is not negative, then it is positive.

6. Complete the following table for the statements in question 5 by indicating whether each statement is true or false.

	Conditional Statement	Inverse	Converse	Contrapositive
a)				
b)				
c)				
d)				

7. Examine your table for question 6.

a) What do you notice about each conditional statement and its contrapositive?
b) What do you notice about the inverse and the converse?

8. Examine your table for question 6 again.

a) What conclusion can you draw about each conditional statement and its converse?
b) What conclusion can you draw about the inverse and the contrapositive?

9. Consider this statement: If a polygon is a square, then the polygon is a quadrilateral.

a) Write the converse, the inverse, and the contrapositive.
b) Verify that each statement is true, or disprove it with a counterexample.

10. Consider this statement: "If the equation of a line is $y = 5x + 2$, then its y-intercept is 2."

a) Write the converse, the inverse, and the contrapositive.
b) Verify that each statement is true, or disprove it with a counterexample.

11. Suppose that a conditional statement, its inverse, its converse, and its contrapositive are all true. What do you know about the conditional statement?

12. For each conditional statement below,
   i) verify it, or disprove it with a counterexample,
   ii) verify the converse, or disprove it with a counterexample,
   iii) verify the inverse, or disprove it with a counterexample, and
   iv) verify the contrapositive, or disprove it with a counterexample.

   a) If the Moon is a balloon, then a pin can burst the Moon.
   b) If x is a negative number, then $-x$ is a positive number.
   c) If a number is a perfect square, then it is positive.
   d) If a number can be expressed as a terminating decimal, then it can be expressed as a fraction.
   e) If the equation of a function is $f(x) = 5x^2 + 10x + 3$, then its graph is a parabola.
   f) If a number is an integer, then it is a whole number.
   g) If I am 18, then I am old enough to vote.
   h) If I am a Canadian, then I enjoy hockey.

Closing

13. Explain, in your own words, why each statement is true.
   a) When a conditional statement is true, its contrapositive will be true.
   b) When the converse of a conditional statement is true, its inverse will be true.

Extending

14. **a)** Write a false conditional statement. Show, by counterexample, that the contrapositive is also false.
   b) Write a true conditional statement. Show that the contrapositive is also true.

15. **a)** Write a conditional statement whose inverse is false. Show, by counterexample, that the converse is also false.
   b) Write a conditional statement whose inverse is true. Show that the converse is also true.

3 Chapter Self-Test

1. The following five Aboriginal writers from across Canada toured Australia. They gave readings and lectures, and they participated in panels and workshops. Categorize these writers using a Venn diagram.
   - Louise Halfe, poet
   - Lee Maracle, poet, novelist
   - Rita Mestokosho, poet
   - Armand Ruffo, poet, fiction writer
   - Richard Van Camp, poet, fiction writer

Louise Halfe was raised on the Saddle Lake First Nation, in Alberta.

Lee Maracle grew up next to the Tsleil-Waututh Nation (Burrard Reserve) in North Vancouver.

Rita Mestokosho is an Innu, born in Ekuanitshit (Mingan) in Québec.

Armand Ruffo is an Ojibway, born in Chapleau, Ontario.

Richard Van Camp is a member of the Tlicho (Dogrib) Nation of the Northwest Territories.

2. Consider these sets:
   $U = \{x \mid x \leq 24, x \in \mathrm{N}\}$ $B = \{x \mid 3x \leq 24, x \in \mathrm{N}\}$
   $A = \{x \mid 2x \leq 24, x \in \mathrm{N}\}$ $C = \{x \mid 4x \leq 24, x \in \mathrm{N}\}$
   a) Show that $C \subset A$ using a Venn diagram.
   b) Determine $A \cup B$, $n(A \cup B)$, $A \cap C$, and $n(A \cap C)$.
   c) Determine $A \cap B \setminus C$.
   d) Determine $(A \cup B \cup C)'$.

3. A survey of students in a school cafeteria had these results:
   - 50% of the students drink bottled water.
   - 56% eat fruit.
   - 43% follow a low-fat diet.
   - 22% drink bottled water and follow a low-fat diet.
   - 23% follow a low-fat diet and eat fruit.
   - 27% drink bottled water and eat fruit.
   - 15% drink bottled water, follow a low-fat diet, and eat fruit.

   What percent of the students do not drink bottled water, do not follow a low-fat diet, and do not eat fruit?

4. Write each statement in "if p, then q" form. Then write the inverse. Determine whether the statements are biconditional, and explain your reasoning. If a statement is biconditional, write it as a biconditional statement. If it is not, state a counterexample.
   a) To win an election, you must get the most votes.
   b) Earth is the third planet from the Sun.
   c) Any number between 1 and 2 is not a whole number.

5. For each conditional statement below,
   i) verify it, or disprove it with a counterexample,
   ii) verify the converse, or disprove it with a counterexample,
   iii) verify the inverse, or disprove it with a counterexample, and
   iv) verify the contrapositive, or disprove it with a counterexample.

   a) If you are over 18, then you are an adult.
   b) If you are 16, then you can drive.

WHAT DO You Think Now? Revisit **What Do You Think?** on page 145. How have your answers and explanations changed?

Chapter Review

FREQUENTLY ASKED Questions

Study Aid

• See Lesson 3.4, Example 2.

Q: How can an understanding of sets help you conduct Internet searches?

A: Sets that are created by using the words "and," "or," or "not" can help you define your search. If you want the exact wording, use quotation marks around the word or phrase. To further reduce your hits, enter a "–" (minus sign) immediately before any words or phrases you want to avoid.

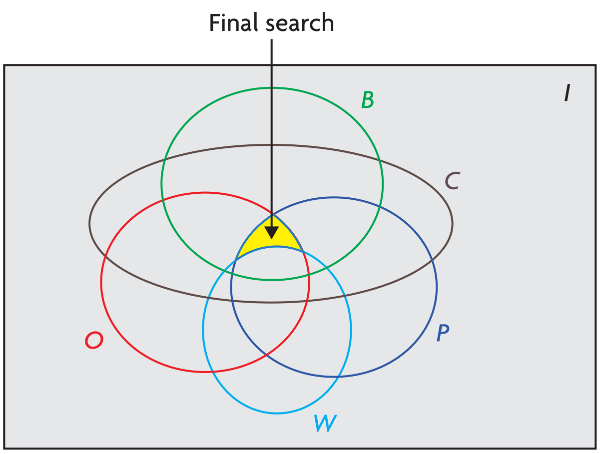
For example, suppose that you want a recipe for chocolate chip cookies like your grandmother used to make. Searching *chocolate chip cookie recipe* nets 938 000 results. Using quotation marks, "*chocolate chip cookie recipe*" gets 172 000 results.

You remember that your grandmother's recipe contained oatmeal and pecans, so you search *"chocolate chip cookie recipe" and oatmeal and pecans*. You get 124 000 results.

You dislike walnuts, so now you search for "*chocolate chip cookie recipe" and oatmeal and pecans –walnuts*. You get 6270 results.

To further narrow your search, you look for the exact phrase *"brown sugar": "chocolate chip cookie recipe" and oatmeal and pecans –walnuts and "brown sugar."* Now you have only 4660 recipes to check.

I = {all search results}
C = {"chocolate chip cookie recipe" }
O = {with oatmeal}
P = {with pecan}
W = {with walnut}
B = {"brown sugar"}



Q: How do you form the converse, inverse, and contrapositive of a conditional statement?

A: In the following table, p represents a hypothesis and q represents a conclusion.

Study Aid

- See Lesson 3.5, Examples 2 and 4; Lesson 3.6.

Type of Statement	Conditional statement	Converse	Inverse	Contrapositive
How to Create Statement	The truth of p implies the truth of q.	Switch p and q.	Negate p and q.	Negate and switch p and q.
Written in Logic Notation	$p \Rightarrow q$	$q \Rightarrow p$	$\neg p \Rightarrow \neg q$	$\neg q \Rightarrow \neg p$
Example	If a bird quacks, then it is a duck.	If a bird is a duck, then it quacks.	If a bird does not quack, then it is not a duck.	If a bird is not a duck, then it does not quack.

Q: How do you decide whether a conditional statement is true or false?

A: Assume that the hypothesis is true. You then need to determine if the conclusion that follows is true or false. If the conclusion is true, then the conditional statement is true.

If the conclusion is false, then the conditional statement is false.

You can also use this approach to decide if the converse, inverse, or contrapositive statements are true.

Note the following:

- If the hypothesis is false, then the conditional statement is always true.
- If a conditional statement is true, then its contrapositive is true, and vice versa.
- If the converse of a conditional statement is true, then its inverse is true.

Study Aid

- See Lesson 3.5, Examples 1 and 2; Lesson 3.6, Examples 1 to 3.
- Try Chapter Review Question 8.

Q: What is a biconditional statement, and how can you create it?

A: A biconditional statement is a statement in which both the original conditional statement and its converse are always true. Combine both the statement and its converse, using "if and only if."
For example: A number is divisible by 2, if and only if it is even.

Study Aid

- See Lesson 3.5, Examples 3 to 5; Lesson 3.6, Example 3.
- Try Chapter Review Question 6.

PRACTISING

Lesson 3.1

1. a) Draw a Venn diagram to show these sets.
- the universal set $U = \{\text{natural numbers from } 1 \text{ to } 30 \text{ inclusive}\}$
- $E = \{\text{multiples of } 2\}$
- $F = \{\text{multiples of } 16\}$
- $S = \{\text{multiples of } 3\}$

b) List the disjoint subsets, if there are any.
c) Is any set a subset of the other sets? Explain.
d) Define S'. How is it different from E'?
e) Give an example of an empty set.

Lesson 3.2

2. There are 28 students on the school track and field team.
- 19 have black hair.
- 8 have blue eyes.
- 9 do not have black hair or blue eyes.

a) How many students have black hair and blue eyes? Explain.
b) How many students have black hair but not blue eyes?
c) How many students have blue eyes but not black hair?

Lesson 3.3

3. Consider these two sets:
- $A = \{-12, -9, -6, -3, 0, 3, 6, 9, 12\}$
- $B = \{x \mid -12 \leq x \leq 12, x \in I\}$

a) Determine $A \cup B$, $n(A \cup B)$, $A \cap B$, and $n(A \cap B)$.
b) Draw a Venn diagram to show these two sets.

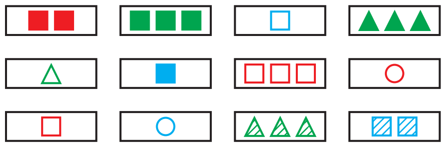
4. Neil asked 40 people at a bookstore if they prefer romance novels or horror novels.
- 18 people do not like either type.
- 10 people like romance novels.
- 13 people like horror novels.

Determine how many people like both romance novels and horror novels.

Lesson 3.4

5. The following 12 cards have three different shapes, colours, and numbers. Create six sets, with three cards in each set. Each set of three cards must have
- the same number or three different numbers, and
- the same shape or three different shapes, and
- the same colour or three different colours.

All the cards in each set can be used more than once.



Lesson 3.5

6. Determine whether each statement is biconditional, and explain your reasoning. If the statement is biconditional, write it in biconditional form. If it is not biconditional, give a counterexample.
a) If x is positive, then $10x > x$.
b) If you live in Victoria, then you live on Vancouver Island.
c) If xy is an odd number, then both x and y are odd numbers.
d) If two numbers are even, then their sum is even.

7. Serge is considering buying a new car but must borrow $24 729.56 from the dealership for 60 months, at 2.9% interest.
a) If he buys the car, then what will his monthly payment be?
b) If Serge chose to pay an additional $100 a month, then how much sooner could he pay off the loan?

Lesson 3.6

8. For each conditional statement below, verify or disprove the statement, its converse, its inverse, and its contrapositive with a counterexample.
a) If a number is positive, then it is not negative.
b) If Monday is a holiday, then it is a long weekend.